Uncertain bilevel knapsack problem and its solution

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Abstract: This paper aims at providing an uncertain bilevel knapsack problem (UBKP) model, which is a type of BKP's involving uncertain variables. And then an uncertain solution for the UBKP is proposed by defining $\mathcal{P}_E$ Nash equilibrium and $\mathcal{P}_E$ Stackelberg Nash equilibrium. In order to improve the computational efficiency of the uncertain solution, several operators (binary coding distance, inversion operator, explosion operator and binary back learning operator) are applied to the basic fireworks algorithm to design the binary backward fireworks algorithm (BBFWA), which has a good performance in solving the BKP. As an illustration, a case study of the UBKP model and the $\mathcal{P}_E$ uncertain solution is applied to an armaments transportation problem.

Keywords: uncertainty, bilevel programming, knapsack problem, binary backward fireworks algorithm.

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1. Introduction

The bilevel knapsack problem (BKP) is a typical bilevel programming which models the hierarchical relationship between two autonomous, and possibly conflicting, decision makers: the leader and the follower [1]. BKP is first studied by Dempe and Richter [2], and it is formulated as a mixed-integer bilevel programming. Recently, more and more researchers focused on the BKP and developed several kinds of algorithm to solve the BKP, such as branch-and-cut [1,2], approximation algorithm [3], dynamic programming [4,5], heuristic optimization algorithm [6,7]. However, almost all of them are studied under the condition of deterministic assumptions.

However, parameters in the BKP are often indeterminism. Traditionally, the probability theory and fuzzy theory are applied to solve indeterminism [8]. As we know, a premise of applying probability theory is that the obtained probability distribution is close enough to the true frequency. However, the plenty of practical sample data to

generate the fitting probability distribution are always impossible. And then, the fuzzy theory may be applied by inviting some domain experts to evaluate the belief degree that each event happens. However, a lot of surveys show that human beings usually estimate a much wider range of values than the object actually takes. Considering all these limitations, Liu declared that it was inappropriate to apply both the probability theory and fuzzy theory to handle uncertainty, for both theories may lead to counterintuitive results [9].

Mostly, the scale of sample data is always too small to generate the probability distribution, but sufficient to obtain the belief degree of these quantities. These kinds of indeterminacy are neither like randomness nor like fuzziness, it is uncertain which is ubiquitous in reality [10], especially issues related to combat, such as the armaments transportation problem. In order to deal with problems under the uncertain condition, Liu found the uncertainty theory, which provides the commonness of the probability theory, credibility theory and chance theory to handle uncertainty [11]. Parameters of the BKP are always uncertain variables, and these kinds of BKP's are defined as the uncertain BKP (UBKP).

In order to deal with the UBKP, an uncertain solution is proposed by presenting $\mathcal{P}_E$ Nash equilibrium and $\mathcal{P}_E$ Stackelberg Nash equilibrium to transform the UBKP to the corresponding equivalent deterministic model. Considering the BKP is NP-hard [12], a heuristic optimization algorithm should be applied to improve computational efficiency. As a relatively new member of swarm intelligence, the fireworks algorithm (FWA) is proposed and enhanced [13,14]. Due to its simplicity, effectiveness and efficiency, FWA has been adopted by researchers in a variety of fields, including variable-rate fertilization in oil crop production [15], and digital filters design [16]. Considering the combinatorial character of the UBKP, a novel evolutionary algorithm named binary backward FWA (BBFWA) is designed, which uses three binary definitions (binary coding distance, inversion operator and explosion operator) to

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transform the basic FWA into binary programming, and applies the binary backward learning operator to improve exploration ability of the FWA.

This paper is organized as follows. In Section 2, some axiom, definition and theorem of the uncertain theory which may be used in the UBKP are introduced. In Section 3, the formulation of the UBKP and its uncertain solution are proposed. Section 4 is devoted to design the BBFWA and test its effectiveness. In Section 5, a case study is provided. Conclusions are stated in Section 6.

2. Preliminaries

In order to rationally deal with belief degree and uncertain problems, several foundation definitions are introduced as follows.

Let Γ be a nonempty set, and let $\mathcal{L}$ be a σ -algebra over Γ . Each element $\wedge \in \mathcal{L}$ is called an event. In order to propose the belief degree that $\wedge$ may occur, uncertain measure is defined as a function from $\mathcal{L}$ to $[0, 1]$, which assigns a real number $\mathcal{M}\{\wedge\}$ to event $\wedge$. And the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Definition 1 Uncertain variable [10]

An uncertain variable is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any set B of real numbers.

Definition 2 Independence [17]

The uncertain variables $\xi_1, \xi_2, \dots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^n (\xi_i \in B_i)\right\} = \min \mathcal{M}(\xi_i \in B_i) \quad (1)$$

for any Borel sets $B_1, B_2, \dots, B_n$ of real numbers.

Definition 3 Uncertainty distribution [10]

The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\} \quad (2)$$

for any real number x .

Two basic forms of uncertainty distribution are shown as follows.

(i) Linear uncertainty distribution $\mathcal{L}(a, b)$ is defined by

$$\Phi(x) = \begin{cases} 0, & x \leq a \\ \frac{(x-a)}{(b-a)}, & a \leq x \leq b \\ 1, & x \geq b \end{cases} \quad (3)$$

where a and b are real numbers with $a < b$.

(ii) Zigzag uncertainty distribution $\mathcal{Z}(a, b, c)$ is defined

by

$$\Phi(x) = \begin{cases} 0, & x \leq a \\ \frac{(x-a)}{2(b-a)}, & a \leq x \leq b \\ \frac{(x+c-2b)}{2(c-b)}, & b \leq x \leq c \\ 1, & x \geq c \end{cases} \quad (4)$$

where a, b, c are real numbers with $a < b < c$.

Definition 4 Inverse uncertainty distribution [10]

Let ξ be an uncertain variable with regular uncertainty distribution Φ . Then the inverse function Φ^{-1} is called the inverse uncertainty distribution of ξ .

Theorem 1 [18] Let ξ be an uncertain variable with a regular uncertainty distribution Φ , then

$$E(\xi) = \int_0^1 \Phi^{-1}(\alpha) d\alpha. \quad (5)$$

Theorem 2 [18] Let $\xi_1, \xi_2, \dots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \dots, \Phi_n$ respectively. If the function $f(x_1, x_2, \dots, x_n)$ is strictly increasing with respect to $x_1, x_2, \dots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \dots, x_n$, then $\xi = f(x_1, x_2, \dots, x_n)$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \dots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \Phi_{m+2}^{-1}(1-\alpha), \dots, \Phi_n^{-1}(1-\alpha)). \quad (6)$$

3. Problem formulation

In this section, the UBKP is presented, where all parameters related are uncertain variables. At first, the BKP is introduced, and then the mathematical description of the UBKP is proposed.

3.1 Formulation of BKP

The BKP involves two decision-makers interacting hierarchically. A leader, responsible for a subset of variables, has to explicitly take the reaction of a follower (responsible for another subset of variables) into account in its optimization process. The specificity of this BKP is that the follower problem is a knapsack problem and the leader's constraints involve both leader and follower variables. In other words, BKP is an optimization problem whose feasible set is determined by the set of optimal solutions of a parametric knapsack problem. The BKP can be appropriately modeled as the following linear bilevel problem with integer variables

$$\begin{aligned} \max_{(x,y)} F(x, y) &= d^1 x + d^2 y \\ \text{s.t. } B^1 x + B^2 y &\leq c^1. \end{aligned}$$

$\mathbf{y} = \{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k\}$, where $\mathbf{y}_i$ solves :

$$\begin{cases} \max_{\mathbf{y}_i} f_i(\mathbf{y}_i') = \mathbf{d}_i^3 \mathbf{y}_i' \\ \mathbf{b}_i^1 \mathbf{x} + \mathbf{b}_i^2 \mathbf{y}_i' \leq c_i^2 \end{cases} \quad (7)$$

$\mathbf{d}^1$ and $\mathbf{d}^2$ are the value weights of the decision variables $\mathbf{x}$ and $\mathbf{y}$ in the objective function of the leader, the vector $\mathbf{x} \in \{0, 1\}^{N^1}$ ($\mathbf{y} \in \{0, 1\}^{N^2}$) represents the leader's (the i th follower's) decision variables and N^1 (N_i^2) denotes the number of leader (the i th follower) decision variables, matrices $\mathbf{B}_{(k \times N^1)}^1, \mathbf{B}_{(k \times N^1)}^2$ (vectors $\mathbf{b}_i^1, \mathbf{b}_i^2$) are the constraint coefficients of the decision variables $\mathbf{x}$ and $\mathbf{y}$ in the objective function of the leader (the i th follower), $c^1(c_i^2)$ are leader's (the i th follower's) knapsack capacity. All the coefficients are assumed to be positive.

The BKP is deduced to the standard 0-1 knapsack problem by setting $N^1 = 0$ and $\mathbf{d}^2 = [c_1^2, c_2^2, \dots, c_k^2]$. As a consequence, BKP is NP-hard.

3.2 Formulation of UBKP

The UBKP is a special BKP in which all the parameters involved are uncertain variables. A formal mathematical description of UBKP which is based on formulation of BKP is presented as follows:

$$\begin{aligned} \max_{(\mathbf{x}, \mathbf{y})} F(\mathbf{x}, \mathbf{y}, \xi) &= \delta^1 \mathbf{x} + \delta^2 \mathbf{y} \\ \text{s.t. } G &= \{\mathbf{B}^1 \mathbf{x} + \mathbf{B}^2 \mathbf{y} - \chi^1 \leq 0\} \\ \mathbf{y} &= \{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k\}, \text{ where } \mathbf{y}_i \text{ solves:} \\ &\begin{cases} \max_{\mathbf{y}_i'} f_i(\mathbf{y}_i', \xi) = \delta_i^3 \mathbf{y}_i' \\ g_i = \{\beta_i^1 \mathbf{x} + \beta_i^2 \mathbf{y}_i' - \chi_i^2 \leq 0\} \end{cases} \end{aligned} \quad (8)$$

$\xi = \{\delta^1, \delta^2, \mathbf{B}^1, \mathbf{B}^2, \chi^1, \delta_i^3, \beta_i^1, \beta_i^2, \chi_i^2\}$ are uncertain parameters corresponding to parameters of $\mathbf{d}^1, \mathbf{d}^2, \mathbf{B}^1, \mathbf{B}^2, c^1, \mathbf{b}_i^1, \mathbf{b}_i^2, c_i^2$ in the model (7).

Based on the penalty function method, the model (8) can be simplified as follows:

$$\begin{aligned} \max_{(\mathbf{x}, \mathbf{y})} F(\mathbf{x}, \mathbf{y}, \xi) &= \rho^1 \cdot (\delta^1 \mathbf{x} + \delta^2 \mathbf{y}) \\ \mathbf{y} &= \{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k\}, \text{ where } \mathbf{y}_i \text{ solves:} \\ \max_{\mathbf{y}_i'} f_i(\mathbf{y}_i', \xi) &= \rho_i^2 \cdot (\delta_i^3 \mathbf{y}_i'). \end{aligned} \quad (9)$$

ρ^1 and ρ_i^2 are penalty factors which are denoted as follows:

$$\rho^1 = \begin{cases} 1, & C^1 = \{(\mathbf{x}, \mathbf{y}) \in G\} \\ 0, & \text{else} \end{cases} \quad (10)$$

$$\rho_i^2 = \begin{cases} 1, & C_i^2 = \{(\mathbf{x}, \mathbf{y}) \in g_i\} \\ 0, & \text{else} \end{cases} \quad (11)$$

3.3 Uncertain solution for UBKP

In order to solve the UBKP on the uncertain objectives directly, definitions of relationship between different uncertain variables are presented first.

Definition 5 Expected-value principle $\mathcal{P}_E$ [19]

Let ξ and η be two uncertain variables, we say $\xi \preceq$ (or $\prec$) η if and only if $E[\xi] \leq$ (or $<$) $E[\eta]$, where $E[\cdot]$ denotes the expected value of the uncertain variable.

Definition 6 Uncertain-constraint principle

Let $g^1(\xi^1) \preceq \eta^1$ be the uncertain constraint condition of the leader knapsack and $g_i^2(\xi_i^2) \preceq \eta_i^2$ be the uncertain constraint condition of the i th follower knapsack in the BKP, and α^1, α_i^2 be the confidence levels of the leader knapsack, the i th follower knapsack. Then we have chance constraints of the leader knapsack and the i th follower knapsack as follows:

$$\mathcal{M}\{g^1(\xi^1) \preceq \eta^1\} \geq \alpha^1, \quad (12)$$

$$\mathcal{M}\{(g^1(\xi^1) \preceq \eta^1) \cap (g_i^2(\xi_i^2) \preceq \eta_i^2)\} \geq \alpha_i^2. \quad (13)$$

Remark 1 Confidence level α means that there is a probability of at least 95% that the result is reliable.

Remark 2 According to Definition 2, the i th follower constraint could be transformed as follows when the uncertain constraint condition of the leader knapsack $g^1(\xi^1) \preceq \eta^1$ and the uncertain constraint condition of the i th follower knapsack $g_i^2(\xi_i^2) \preceq \eta_i^2$ are mutually independent

$$\min(\mathcal{M}(g^1(\xi^1) \preceq \eta^1), \mathcal{M}(g_i^2(\xi_i^2) \preceq \eta_i^2)) \geq \alpha_i^2. \quad (14)$$

Since the objective functions are uncertain variables and the constraint conditions are uncertain inequality constraints, the problem cannot be directly maximized. Instead, we maximize its expected value subject to a set of chance constraints:

$$\max_{(\mathbf{x}, \mathbf{y})} E[F(\mathbf{x}, \mathbf{y}, \xi)] = \rho^1 \cdot E[\delta^1 \mathbf{x} + \delta^2 \mathbf{y}]$$

$$\mathbf{y} = \{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k\}, \text{ where } \mathbf{y}_i \text{ solves:}$$

$$\max_{\mathbf{y}_i'} E[f_i(\mathbf{y}_i', \xi)] = \rho_i^2 \cdot E[\delta_i^3 \mathbf{y}_i']. \quad (15)$$

ρ^1 and ρ_i^2 are penalty factors under the uncertain-constraint principle, which are denoted as follows:

$$\rho^1 = \begin{cases} 1, & \mathcal{M}\{C^1\} \geq \alpha^1 \\ 0, & \text{else} \end{cases}, \quad (16)$$

$$\rho_i^2 = \begin{cases} 1, & \mathcal{M}\{C^1 \cap C_i^2\} \geq \alpha_i^2 \\ 0, & \text{else} \end{cases}. \quad (17)$$

Definition 7 $\mathcal{P}_E$ Nash equilibrium

Let $\mathbf{x}$ be a feasible decision vector of the leader. A $\mathcal{P}_E$ Nash equilibrium of followers is defined as the feasible array $(\mathbf{y}_1^*, \mathbf{y}_2^*, \dots, \mathbf{y}_k^*)$ with respect to $\mathbf{x}$ if there is no feasible array $(\mathbf{y}_1^*, \mathbf{y}_2^*, \dots, \mathbf{y}_{i-1}^*, \mathbf{y}^*, \mathbf{y}_{i+1}^*, \dots, \mathbf{y}_k^*)$ such that

$$E \left[f_i \left(\mathbf{x}, \mathbf{y}_1^*, \mathbf{y}_2^*, \dots, \mathbf{y}_{i-1}^*, \mathbf{y}^*, \mathbf{y}_{i+1}^*, \dots, \mathbf{y}_k^*, \xi \right) \right] \geq$$

$$\mathbb{E} \left[f_i \left(\mathbf{x}, \mathbf{y}_1^*, \mathbf{y}_2^*, \dots, \mathbf{y}_{i-1}^*, \mathbf{y}_i^*, \mathbf{y}_{i+1}^*, \dots, \mathbf{y}_k^*, \boldsymbol{\xi} \right) \right]. \quad (18)$$

Definition 8 $\mathcal{P}_E$ Stackelberg Nash equilibrium

Let $\mathbf{x}^*$ be a feasible decision vector of the leader and $(\mathbf{y}_1^*, \mathbf{y}_2^*, \dots, \mathbf{y}_k^*)$ be $\mathcal{P}_E$ Nash equilibrium of followers with respect to $\mathbf{x}^*$. A $\mathcal{P}_E$ Stackelberg Nash equilibrium to the bilevel problem is defined as $(\mathbf{x}^*, \mathbf{y}_1^*, \mathbf{y}_2^*, \dots, \mathbf{y}_k^*)$ if there is no feasible array $(\mathbf{x}^{**}, \mathbf{y}_1^{**}, \mathbf{y}_2^{**}, \dots, \mathbf{y}_k^{**})$ such that

$$\mathbb{E}[F(\mathbf{x}^{**}, \mathbf{y}_1^{**}, \mathbf{y}_2^{**}, \dots, \mathbf{y}_k^{**}, \boldsymbol{\xi})] \geq \mathbb{E}[F(\mathbf{x}^*, \mathbf{y}_1^*, \mathbf{y}_2^*, \dots, \mathbf{y}_k^*, \boldsymbol{\xi})] \quad (19)$$

where $(\mathbf{y}_1^{**}, \mathbf{y}_2^{**}, \dots, \mathbf{y}_k^{**})$ is the $\mathcal{P}_E$ Nash equilibrium of followers with respect to $\mathbf{x}^{**}$.

3.4 Equivalent deterministic model for UBKP

From the mathematical viewpoint, the UBKP model cannot be solved directly. It is essential to transform it into the corresponding equivalent deterministic BKP model.

Let $\xi_1, \xi_2, \dots, \xi_m$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \dots, \Phi_m$ respectively. Without loss of generality, we assume that the leader

function F is a real function which is strictly increasing with respect to $\xi_1, \xi_2, \dots, \xi_j$ and strictly decreasing with respect to $\xi_{j+1}, \xi_{j+2}, \dots, \xi_m$. According to Theorem 1 and Theorem 2, we have

$$\mathbb{E}[F(\mathbf{x}, \mathbf{y}, \xi_1, \xi_2, \dots, \xi_m)] = \rho^1 \cdot \int_0^1 F(\mathbf{x}, \mathbf{y}, \Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \dots, \Phi_j^{-1}(\alpha), \Phi_{j+1}^{-1}(1-\alpha), \dots, \Phi_m^{-1}(1-\alpha)) d\alpha. \quad (20)$$

Similarly, assume that the i th follower function f_i is a real function which is strictly increasing with respect to $\xi_1, \xi_2, \dots, \xi_{j_i}$ and strictly decreasing with respect to $\xi_{j_i+1}, \xi_{j_i+2}, \dots, \xi_m$, the leader constraint function G is a real function which is strictly increasing with respect to $\xi_1, \xi_2, \dots, \xi_{j_G}$ and strictly decreasing with respect to $\xi_{j_G+1}, \xi_{j_G+2}, \dots, \xi_m$ and the i th follower function g_i is a real function which is strictly increasing with respect to $\xi_1, \xi_2, \dots, \xi_{j_{g_i}}$ and strictly decreasing with respect to $\xi_{j_{g_i}+1}, \xi_{j_{g_i}+2}, \dots, \xi_m$. The UBKP model can be transformed into the equivalent deterministic BKP model as follows:

$$\begin{aligned} & \max_{(\mathbf{x}, \mathbf{y})} \left(\rho^1 \cdot \int_0^1 F(\mathbf{x}, \mathbf{y}, \Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \dots, \Phi_j^{-1}(\alpha), \Phi_{j+1}^{-1}(1-\alpha), \dots, \Phi_m^{-1}(1-\alpha)) d\alpha \right) \\ & \rho^1 = \begin{cases} 1, & \int_0^1 G(\mathbf{x}, \mathbf{y}, \Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \dots, \Phi_{j_G}^{-1}(\alpha), \Phi_{j_G+1}^{-1}(1-\alpha), \dots, \Phi_m^{-1}(1-\alpha)) d\alpha \leq 0 \\ 0, & \text{else} \end{cases} \\ & \mathbf{y} = \{\mathbf{y}_1, \mathbf{y}_1, \dots, \mathbf{y}_1\}, \text{ where } \mathbf{y}_i \text{ solves :} \\ & \begin{cases} \max_{(\mathbf{x}, \mathbf{y})} \left(\rho_i^2 \cdot \int_0^1 f_i(\mathbf{x}, \mathbf{y}, \Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \dots, \Phi_{j_i}^{-1}(\alpha), \Phi_{j_i+1}^{-1}(1-\alpha), \dots, \Phi_m^{-1}(1-\alpha)) d\alpha \right) \\ \rho^1 = \begin{cases} 1, & \left(\int_0^1 g_i(\mathbf{x}, \mathbf{y}, \Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \dots, \Phi_{j_{g_i}}^{-1}(\alpha), \Phi_{j_{g_i}+1}^{-1}(1-\alpha), \dots, \Phi_m^{-1}(1-\alpha)) d\alpha \leq 0 \right) \cap (\rho^1 = 1) \\ 0, & \text{else} \end{cases} \end{cases} \end{aligned} \quad (21)$$

4. BBFWA for UBKP

In order to solve the UBKP, a numerical method is needed to improve the computational efficiency. In this section, a new evolutionary algorithm named BBFWA is developed.

4.1 Design of BBFWA

BBFWA space is an $n \times D$ Euclidean space which is made up by n fireworks devices and D -dimensional binary coding items (arguments space of optimization function). Location of element (fireworks, sparks or mutation sparks) i is $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \dots, x_{i,D})$ and gorgeous degree (objective function value) of the element i is $Y_i = f(\mathbf{x}_i)$.

BBFWA is constituted of binary coding distance, inversion operator, backward operator, explosion operator, fireworks explosion, mutation explosion and selection strategy.

(i) Binary coding distance

Binary coding distance is proposed to describe the difference between two binary items, and the binary coding distance between $\mathbf{x}_i$ and $\mathbf{x}_j$ is defined as

$$d(\mathbf{x}_i, \mathbf{x}_j) = \sum_{k=1}^D |x_{i,k} - x_{j,k}| \quad (22)$$

where D is the dimension of $\mathbf{x}_i$ and $\mathbf{x}_j$.

(ii) Inversion operator

Inversion operator is provided to invert the characters of the binary code 0 to 1 (or 1 to 0) effectively (i.e., to change the status of the item k whether it is in the knapsack), which is defined as follows:

$$\mathcal{J}(x_{i,k}) = 1 - x_{i,k}. \quad (23)$$

(iii) Explosion operator

Explosion operator is proposed to describe the explosion of fireworks. Let $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \dots, x_{i,D})$ be the position, S be the explosion space of the element i (i.e., a set of the binary characters that may be inverted) and $Step$ be the step of explosion (i.e., the number of the binary characters that must be inverted). Explosion operator $\mathcal{N}(\mathbf{x}_i, S, r)$ means that r binary character which is randomly selected from the set S is inverted by the inversion operator.

(iv) Binary backward learning operator

In many cases, the optimization begins at random items. The random guess, if not far away from the optimal solution, can result in a fast convergence. However, the random guess may be very far away from the existing solution, even in the opposite location. Then the optimization takes considerably more time, even becomes intractable. Thus, searching in the opposite direction could be beneficial, and the opposite number of binary coding items is defined as follows.

Definition 9 Binary opposite items

Let $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \dots, x_{i,D})$ be a D -dimensional binary items. The binary opposite number is defined as $\tilde{\mathbf{x}}_i = (\tilde{x}_{i,1}, \tilde{x}_{i,2}, \dots, \tilde{x}_{i,D})$, where $\tilde{x}_{i,k}$ is the k th binary opposite character which is defined as

$$\tilde{x}_{i,k} = \mathcal{J}(x_{i,k}) = 1 - x_{i,k}. \quad (24)$$

Assume that $\mathbf{x}_i$ is one of the random initial (random) items and $f(\cdot)$ is a proper evaluation function. In each iteration, the value of items $\mathbf{x}_i$ and the value of its opposite number $\tilde{\mathbf{x}}_i$ is obtained, and then the evolution continues with $\mathbf{x}_i$ if $f(\mathbf{x}_i) \geq f(\tilde{\mathbf{x}}_i)$, otherwise with $\tilde{\mathbf{x}}_i$

$$\mathbf{x}_i = \begin{cases} \mathbf{x}_i, & f(\mathbf{x}_i) \geq f(\tilde{\mathbf{x}}_i) \\ \tilde{\mathbf{x}}_i, & \text{else} \end{cases}. \quad (25)$$

(v) Fireworks explosion

Fireworks explosion generates several sparks to explore a new location around the explosion location, which can be viewed as a searching process around a specific point in each generation. High-quality fireworks may generate numerous sparks and the sparks centralize the explosion center, just as spot near the optimal location needs to be well searched. However, low-quality fireworks may generate quite few sparks and the sparks scatter in the space.

In order to obtain splendid fireworks explosion, it is wise to set off different quality fireworks to the spot according to its gorgeous degree (objective function value). Then, the explosion amplitude A_i and the number of explosion sparks N_i generated by fireworks i is expressed as follows:

$$A_i = \text{ceil} \left(D \cdot \frac{f_{\max} - f(\mathbf{x}_i) + \varepsilon}{\sum_{i=1}^n (f_{\max} - f(\mathbf{x}_i)) + \varepsilon} \right) \quad (26)$$

$$N_i = \text{ceil} \left(N_C \cdot \frac{f(\mathbf{x}_i) - f_{\min} + \varepsilon}{\sum_{i=1}^n (f(\mathbf{x}_i) - f_{\min}) + \varepsilon} \right) \quad (27)$$

where $\text{ceil}(\cdot)$ maps a real number to the smallest following integer not less than input variable, D is the dimension of items $\mathbf{x}_i$, $f_{\max}(f_{\min})$ is the maximum (minimum) value of the objective function among n fireworks, N_C is a constant to control the number of explosion sparks and ε is the machine epsilon which guarantees A_i and N_i are both meaningful.

And then, items of the p th spark generated from fireworks $\mathbf{x}_i$ is defined as

$$\mathbf{x}_i^p = \mathcal{N}(\mathbf{x}_i, S_0, r_i^E) \quad (28)$$

where $S_0 = \{1, 2, \dots, D\}$ is the set with all the serial number of binary characters, r_i^E is the fireworks explosion step coefficient which is defined as follows:

$$\text{Step}_i^E = \text{ceil}(A_i \cdot \text{rand} + \varepsilon). \quad (29)$$

(vi) Mutation explosion

In order to keep the diversity of sparks, the mutation explosion is designed to generate mutation sparks which escape from the best location $\mathbf{x}_{\text{best}}$ as much as possible. Thus the mutation space is defined by a set with all the serial number of characters which are the same as the best location $\mathbf{x}_{\text{best}}$. The mutation space of fireworks i is defined by $S_i^m = \{s_1, s_2, \dots, s_D\}$, in which

$$s_j = \begin{cases} j, & x_{i,j} = x_{\text{best},j} \\ \text{null}, & \text{else} \end{cases} \quad (30)$$

where null is a null character.

Contrary to the fireworks explosion, sparks generated by high-quality fireworks may be scattered in the space to escape from father fireworks. Mutation amplitude is defined

as

$$A_i^M = \text{ceil} \left(D \cdot \frac{f(\mathbf{x}_i) - f_{\min} + \varepsilon}{\sum_{i=1}^n (f(\mathbf{x}_i) - f_{\min}) + \varepsilon} \right). \quad (31)$$

Similar to the step of the fireworks explosion coefficient, the step of the mutation explosion is obtained as follows:

$$r_i^M = \text{ceil}(A_i^M \cdot \text{rand} + \varepsilon). \quad (32)$$

And then, mutation sparks generated from fireworks $\mathbf{x}_i$ are defined as

$$\mathbf{x}_i^M = \mathcal{N}(\mathbf{x}_i, S_i^M, r_i^M). \quad (33)$$

(vii) Selection strategy

After fireworks explosion and mutation explosion, locations of sparks and mutation sparks have been well searched as well as the location of fireworks. To select the fireworks for the next generation, the best element is always kept at first. Then the other $n - 1$ fireworks are selected based on their distance to other elements so as to keep diversity. Element that is far from other elements gets more chance to be selected than those elements with the smaller distances to other elements. The probability of whether the element i should be selected to be fireworks in the next generation is expressed as follows:

$$p_i = \frac{\sum_{j \in K} d(\mathbf{x}_i, \mathbf{x}_j)}{\sum_{i \in K} \sum_{j \in K} d(\mathbf{x}_i, \mathbf{x}_j)} \quad (34)$$

where K is the set of all elements (fireworks, sparks and mutation sparks).

4.2 Framework of BBFWA for UBKP

The framework of the BBFWA is illustrated in Fig. 1. The binary backward learning operator is devoted to start the fireworks explosion from the efficient location in each generation. The fireworks explosion is proposed to obtain better solutions in the local exploitation. The mutation explosion is applied to improve the searching efficiency in the global exploration. In the selection strategy, the elite strategy is applied to keep the elitism selection, and the probability selection based on the binary coding distance is used to keep diversity of population.

And then, the BBFWA is applied to solve the UBKP. Based on the uncertain solution introduced in Section 3.3 and equivalent deterministic model in Section 3.4, the framework of BBFWA for UBKP is proposed as follows.

Fig. 1 Framework of BBFWA

Step 1 Set parameters of the BBFWA, such as the population of fireworks, the maximum number of iterations, the maximum number of explosion sparks and the number of mutation sparks.

Step 2 Initialize fireworks of the leader knapsack in the feasible set randomly.

Step 3 Generate binary opposite items of fireworks.

Step 4 Act the binary backward learning operator to generate efficient fireworks.

Step 5 Determine the $\mathcal{P}_E$ Nash equilibrium of followers with respect to each leader solution items via the BBFWA.

Step 6 Calculate values of the leader knapsack for each solution items based on its $\mathcal{P}_E$ Nash equilibrium.

Step 7 Select elements to be fireworks of the next generation by the selection strategy.

Step 8 Repeat Step 3 to Step 7 till evolutionary cycles meet the maximum iteration.

Step 9 Choose the best element as the optimal solution, which is the $\mathcal{P}_E$ Stackelberg Nash equilibrium of the UBKP.

4.3 Analysis of BBFWA on deterministic BKP

In order to illustrate the effectiveness of the BBFWA in solving the UBKP, the simulation on deterministic BKP benchmark [20] is carried out for the UBKP is a special BKP which is under uncertain condition.

Assume there is one leader and one follower in the BKP model whose control vectors are $\mathbf{x} = [x_1, x_2, x_3]^T$ and $\mathbf{y} = [y_1, y_2, y_3, y_4]^T$ respectively. Suppose that the BKP model is formulated as follows:

$$\begin{aligned} \max_{(\mathbf{x}, \mathbf{y})} F(\mathbf{x}, \mathbf{y}) &= \mathbf{d}^1 \mathbf{x} + \mathbf{d}^2 \mathbf{y} \\ \text{s.t. } \mathbf{x} &\in \{0, 1\}^3 \\ &\begin{cases} \max_{(\mathbf{y})} F(\mathbf{y}) = \mathbf{d}^3 \mathbf{y} \\ \mathbf{b}^1 \mathbf{x} + \mathbf{b}^2 \mathbf{y} \leq c^2 \\ \mathbf{y} \in \{0, 1\}^4 \end{cases} \end{aligned} \quad (35)$$

where $\mathbf{d}^1 = [3, 4, 7]$, $\mathbf{d}^2 = [5, 1, 2, 7]$, $\mathbf{d}^3 = [2, 2, 3, 4]$, $\mathbf{b}^1 = [3, 1, 2]$, $\mathbf{b}^2 = [1, 2, 1, 4]$ and $c^2 = 4$.

Set the population coefficient as four, sparks coefficient as 1 and mutation coefficient as 0.1. Simulation of the BBFWA with 50 generations shows that the Stackelberg-Nash equilibrium are $\mathbf{x} = [0, 0, 1]^T$, $\mathbf{y} = [1, 0, 1, 0]^T$, and the expected objective values of the leader and the follower are 14 and 5, respectively. Note that the Stackelberg-Nash equilibrium is not unique. For example, another Stackelberg-Nash equilibrium is $\mathbf{x} = [0, 1, 1]^T$, $\mathbf{y} = [0, 0, 1, 0]^T$, and the objective values of the leader and the follower are 13 and 3, respectively. And the decision $\mathbf{x} = [0, 0, 1]^T$, $\mathbf{y} = [1, 0, 0, 0]^T$ is the optimal solution for its maximum value of the leader.

In addition, in order to get better performance of the BBFWA on the BKP, parameters setting is investigated systematically based on full factorial experimental design and analysis of variance. When the population coefficient β_p , sparks coefficient β_s and mutation coefficient β_m vary, iterations of the leader I_l (the follower I_f) need to obtain its corresponding optimal solution while the BBFWA also varies. And the values are shown in Table 1. In order to keep the simulation are carried under the same computational cost, we set β_p times β_s to be a constant.

Table 1 Parameters tuning of BBFWA

Number	β_p	β_s	β_m	$I_l \times I_f$
1	3	4/3	0.05	791.26
2	3	4/3	0.10	746.74
3	3	4/3	0.20	737.08
4	4	1	0.05	691.87
5	4	1	0.10	627.69
6	4	1	0.20	623.16
7	5	4/5	0.05	751.92
8	5	4/5	0.10	698.11
9	5	4/5	0.20	684.37

As shown in Table 1, the combination of $\beta_p \times \beta_s$ has a large impact on the computational result. With the limited constant of four, the best result is achieved when the combination is set as 4×1 . Since the mutation coefficient β_m can provide sufficient diversity and improvement in the solution evolution, the performance of the BBFWA does not change significantly when β_m increases. However, the low limit value may worsen the performance of the BBFWA. As a result, we set parameters as $\beta_p = 4$, $\beta_s = 1$ and $\beta_m = 0.1$.

5. Applications

In this section, the UBKP is applied to an armaments transportation problem. Consider that several armaments and combat groups need to be transported to the military front. Assume that the armaments and warrior soldiers supply two functions, such as combat capacity in the frontier and protective capacity in the transportation process. The frontier commander makes a decision on the allocation of the resources to maximize the combat capacity in the frontier, and the transportation commander desires to guarantee the safety in the transportation process. Then an uncertain multilevel programming model of armaments transportation problem is given as follows:

$$\begin{aligned} \max_{(\mathbf{x}, \mathbf{y})} F(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) &= \boldsymbol{\xi}_1 \mathbf{d}^1 \mathbf{x} + \boldsymbol{\xi}_2 \mathbf{d}^2 \mathbf{y} \\ \text{s.t. } \mathbf{x} &\in \{0, 1\}^3 \\ &\begin{cases} \max_{(\mathbf{y})} F(\mathbf{y}, \boldsymbol{\xi}) = \boldsymbol{\xi}_3 \mathbf{d}^3 \mathbf{y} \\ \mathbf{b}^1 \mathbf{x} + \mathbf{b}^2 \mathbf{y} \leq c^2 \\ \mathbf{y} \in \{0, 1\}^4 \end{cases} \end{aligned} \quad (36)$$

where $\mathbf{d}^1 = [3, 4, 7]$, $\mathbf{d}^2 = [5, 1, 2, 1]$, $\mathbf{d}^3 = [3, 2, 2, 4]$, $\mathbf{b}^1 = [3, 1, 2]$, $\mathbf{b}^2 = [1, 2, 1, 4]$, $c^2 = 4$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$, $\xi_1 = \mathcal{L}(0.9, 1.4)$, $\xi_2 = \mathcal{Z}(0.6, 0.8, 1.05)$ and $\xi_3 = \mathcal{Z}(0.8, 0.9, 1.1)$. Notations are introduced as follows, $\mathbf{x} = \{x_i\}_{1 \times n_1}$ are armaments without protective capacity, $\mathbf{y} = \{y_i\}_{1 \times n_2}$ are armaments which have the protective capacity, $\xi_1 \mathbf{d}^1$ ($\xi_1 \mathbf{d}^2$) is the combat capacity factor of armaments $\mathbf{x}(\mathbf{y})$, $\xi_3 \mathbf{d}^3$ is the protective capacity factor

of armaments $\mathbf{y}$, $F(\mathbf{x}, \mathbf{y})$ is the total combat effectiveness and $f(\mathbf{y})$ is the total protective effectiveness.

And then, an uncertain solution based on BBFWA is applied to solve model (36). Results show that the instance has a Stackelberg-Nash equilibrium $\mathbf{x} = [0, 1, 1]^T$, $\mathbf{y} = [0, 0, 1, 0]^T$, and the objective values of the leader and the follower are 14.28 and 2.79, respectively.

The frontier commander requires armament x_2, x_3 to be transported, and the transportation commander sending armament (or group) y_3 to the frontier accompany with the other armaments. In addition, the expected combat capacity of this transportation team may be 14.28 and the expected protective capacity is 2.79.

6. Conclusions

This paper proposes an UBKP model which is a special type of the BKP with uncertain variables. And then an uncertain solution is developed to transform the UBKP into an equivalent deterministic model. Besides, the computational efficiency is improved by using a new designed evolutionary algorithm, BBFWA. And the efficiency of the algorithm is illustrated by deterministic BKP examples. Finally, the UBKP is applied to an armaments transportation problem. As for the future work, we will modify intelligent algorithms and use the UBKP to solve applications of BKP in various areas.

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