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Estimation for constant-stress accelerated life test from generalized half-normal distribution

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Abstract: In the constant-stress accelerated life test, estimation issues are discussed for a generalized half-normal distribution under a log-linear life-stress model. The maximum likelihood estimates with the corresponding fixed point type iterative algorithm for unknown parameters are presented, and the least square estimates of the parameters are also proposed. Meanwhile, confidence intervals of model parameters are constructed by using the asymptotic theory and bootstrap technique. Numerical illustration is given to investigate the performance of our methods.

Keywords: accelerated life test, maximum likelihood estimation, least square method, bootstrap technique, asymptotic distribution.

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1. Introduction

The two-parameter generalized half-normal (GHN) distribution was introduced by Cooray and Ananda [1] and has received wide attention in recent years. The GHN model features decreasing, increasing and bathtub shapes of the hazard function which makes it more flexible and applicable in reliability and lifetime studies. Moreover, property such as positive and negative skewness also makes it have better fitness for real lifetime data.

Let T be a random variable from the GHN distribution, of which the cumulative distribution function (CDF) and probability density function (PDF) with parameters $\theta > 0$ and $\sigma > 0$ are given by

$$F(t) = 1 - 2\Phi(-(t/\sigma)^{\theta}), \quad t > 0$$

$$f(t) = \sqrt{\frac{2}{\pi}}\frac{\theta}{t}\left(\frac{t}{\sigma}\right)^{\theta} \mathrm{e}^{-(t/\sigma)^{2\theta}/2}, \quad t > 0 \qquad (1)$$

where $\Phi(\cdot)$ is the CDF of the standard normal model. The inference on the GHN distribution has been studied by many authors. Ahmadi et al. [2] considered estimation issues for the GHN model under progressive censoring. Olmos et al. [3] proposed an extension of the GHN model and provided some applications to real data examples. Details about the GHN model can be found in [1].

In the reliability analysis, products are usually characterized by high reliability and long span, and it is impractical to test them under the use condition. Fortunately, the accelerated life test (ALT) provides an economical way to obtain failure data rapidly within a reasonable cycle, where units are tested under the higher use stress level. Two common methods in ALTs are constant-stress model and step-stress model, and both models have been studied extensively, for example, the works of Abdel-Hamid [4], Balakrishnan et al. [5], Kateri and Kamps [6], Nelson and Meeker [7]. One can find more details about ALTs in monographs of Nelson [8], Bagdonavicius and Nikulin [9]. As a flexible and applicable lifetime distribution in reliability and lifetime studies, according to our best knowledge, however, there are no studies on the ALT for the GHN model so far. This paper considers the constant-stress ALT when the lifetime of units follows the GHN distribution (1). When the scale model parameter follows the log-linear life-stress model, estimations are discussed for the unknown parameters.

The rest of this paper is organized as follows. Basic assumptions and life test procedure are presented in Section 2. Point and interval estimates are presented in Sections 3 and 4 respectively based on different methods. Numerical illustration is provided in Section 5, and Section 6 presents some concluding remarks.

2. Basic assumptions and life test procedure

2.1 Basic assumptions

The assumptions are made as follows:

(i) The lifetime of units has the GHN distribution at stress s_i $(i = 1, 2)$, of which the PDF is

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$$f_i(t) = \sqrt{\frac{2}{\pi}}\frac{\theta_i}{t}\left(\frac{t}{\sigma_i}\right)^{\theta_i} \mathrm{e}^{-(t/\sigma_i)^{2\theta_i}/2}, \quad t > 0 \tag{2}$$

where parameters θ_i and σ_i affect the geometric shape and the steepness of the model density curve, respectively.

(ii) The shape parameter θ_i is assumed to be the same under stress s_i, i.e.

$$\theta_1 = \theta_2 = \theta,$$

which implies the failure mechanism keeps unchanged at different stress s_i.

(iii) The scale parameter σ_i is a log-linear function of stress s_i, i.e.

$$\ln \sigma_i = \alpha_0 + \alpha_1 s_i, \quad i = 1, 2 \tag{3}$$

where α_0 and α_1 are coefficient parameters.

2.2 Life test procedure

A two-stage constant-stress ALT namely the simple constant-stress model is studied in this paper, and the experiment is described as follows. Under the accelerated stress level s_i $(i = 1, 2)$, there are n_i units running simultaneously and the failure times are recorded until all units fail.

Suppose that $T = \{T_{ij} : j = 1, \cdots, n_i, i = 1, 2\}$ is the observed sample from (2), the likelihood function is given by

$$L(\alpha, \theta) = \prod_{i=1}^{2}\prod_{j=1}^{n_i} f_i(t) \propto \prod_{i=1}^{2} \theta^{n_i}\sigma_i^{-n_i\theta}\cdot$$

$$\exp\left(\theta\sum_{j=1}^{n_i}\ln t_{ij} - \frac{\sigma_i^{-2\theta}}{2}\sum_{j=1}^{n_i}\ln t_{ij}^{2\theta}\right). \tag{4}$$

3. Point estimation

Maximum likelihood estimates (MLEs) and least-square estimates (LSEs) are presented in this section.

3.1 MLE

From (4), the log-likelihood function is

$$l(\alpha, \theta) \propto \sum_{j=1}^{n_i} n_i \ln\theta - \theta\sum_{j=1}^{n_i} n_i \ln\sigma_i +$$

$$\theta\sum_{i=1}^{2}\sum_{j=1}^{n_i}\ln t_{ij} - \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{n_i}\left(\frac{t_{ij}}{\theta}\right)^{2\theta}. \tag{5}$$

Taking the derivative for $l(\alpha, \theta)$ in (5) with respect to σ_i, one has

$$\sigma_i = \left[\frac{1}{n_i}\sum_{j=1}^{n_i} t_{ij}^{2\theta}\right]^{1/2\theta}, \quad i = 1, 2. \tag{6}$$

Substituting (6) to (5), one can obtain the profile log-likelihood function of θ as

$$l(\theta) \propto \sum_{i=1}^{2} P_i(\theta) \tag{7}$$

where

$$P_i(\theta) = n_i\ln\theta + \theta\sum_{j=1}^{n_i}\ln t_{ij} - \frac{n_i}{2}\ln\left(\sum_{j=1}^{n_i} t_{ij}^{2\theta}\right),$$
$$i = 1, 2. \tag{8}$$

Therefore, the MLE of θ can be derived by maximizing (7) with respect to θ.

Lemma 1 Suppose that $P_i(\theta)$ $(i = 1, 2)$ is defined in (8), then $l(\theta)$ in (7) is the concave function.

Proof It is seen that

$$\frac{\partial^2 P_i(\theta)}{\partial\theta^2} = -2n_i\cdot$$

$$\frac{\left[\sum_{j=1}^{n_i} t_{ij}^{2\theta}\ln^2 t_{ij}\right]\left[\sum_{j=1}^{n_i} t_{ij}^{2\theta}\right] - \left[\sum_{j=1}^{n_i} t_{ij}^{2\theta}\ln t_{ij}\right]^2}{\left[\sum_{j=1}^{n_i} t_{ij}^{2\theta}\right]^2} - \frac{n_i}{\theta^2}.$$

Denote quantities $p_i(\theta) = \sum_{j=1}^{n_i} t_{ij}^{2\theta}$ $(i = 1, 2)$, since $p_i''(\theta)p_i'(\theta) \geqslant [p_i(\theta)]^2$ $(i = 1, 2)$, then the second derivative of $P_i(\theta)$ is negative, which implies that $l(\theta)$ is the concave function. □

From Lemma 1 and the fact that $l(\theta) \to -\infty$ as $\theta \to 0$ and $\theta \to \infty$, it is noted that $l(\theta)$ is the unimodal function and has unique maximum. By taking derivative for $l(\theta)$, the MLE of θ can be derived from the following equation:

$$H_1(\theta) - H_2(\theta) = 0 \tag{9}$$

where

$$H_1(\theta) = \frac{\sum_{i=1}^{2} n_i}{\theta}$$

$$H_2(\theta) = \sum_{i=1}^{2}\left[\frac{n_i\sum_{j=1}^{n_i} t_{ij}^{2\theta}\ln t_{ij}}{\sum_{j=1}^{n_i} t_{ij}^{2\theta}} - \sum_{j=1}^{n_i}\ln t_{ij}\right]. \tag{10}$$

Lemma 2 Suppose that $H_1(\theta)$ and $H_2(\theta)$ are defined in (10), then

(i) $H_1(\theta)$ is the decreasing function with

$$\lim_{\theta\to 0} H_1(\theta) = +\infty, \quad \lim_{\theta\to\infty} H_1(\theta) = 0.$$

(ii) $H_2(\theta)$ is the increasing function with

$$\lim_{\theta\to 0} H_2(\theta) = 0,$$

$$\lim_{\theta\to\infty} H_2(\theta) = \sum_{i=1}^{2}\sum_{j=1}^{n_i} [\ln t_{in_i} - \ln t_{ij}] > 0.$$

Proof The results of $H_1(\theta)$ are obvious.

For $H_2(\theta)$, since

$$\frac{\mathrm{d}H_2(\theta)}{\mathrm{d}\theta} = \sum_{i=1}^{2} \frac{2n_i}{\left[\sum\limits_{j=1}^{n_i} t_{ij}^{2\theta}\right]^2}\cdot$$

$$\left(\left[\sum_{j=1}^{n_i} t_{ij}^{2\theta}\ln^2 t_{ij}\right]\left[\sum_{j=1}^{n_i} t_{ij}^{2\theta}\right] - \left[\sum_{j=1}^{n_i} t_{ij}^{2\theta}\ln t_{ij}\right]^2\right).$$

Similar as Lemma 1, one has $\dfrac{\mathrm{d}H_2(\theta)}{\mathrm{d}\theta} > 0$, thus $H_2(\theta)$ is increasing. Moreover, the limitations of $H_2(\theta)$ are obvious that are omitted. □

Based on Lemma 2, there is a unique interaction point for curves of $H_1(\theta)$ and $H_2(\theta)$. From (9), the MLE of θ can be solved from the following fixed type equation

$$H(\theta) = \theta$$

where

$$H(\theta) = \frac{n_1 + n_2}{\sum\limits_{i=1}^{2}\left[\dfrac{n_i\sum\limits_{j=1}^{n_i} t_{ij}^{2\theta}\ln t_{ij}}{\sum\limits_{j=1}^{n_i} t_{ij}^{2\theta}} - \sum\limits_{j=1}^{n_i}\ln t_{ij}\right]}.$$

An iterative procedure is provided to estimate α_0, α_1 and θ as follows.

Iterative algorithm

Step 1 Denote $\theta^{(0)}$ is an initial guess of θ, and obtain $\theta^{(1)} = H(\theta^{(0)})$.

Step 2 In the $(k+1)$th iterative, $\theta^{(k+1)} = H(\theta^{(k)})$, and

$$\alpha_0^{(k+1)} = \frac{s_2\ln(\sigma_1^{(k+1)}) - s_1\ln(\sigma_2^{(k+1)})}{s_2 - s_1},$$

$$\alpha_1^{(k+1)} = \frac{\ln(\sigma_2^{(k+1)}) - \ln(\sigma_1^{(k+1)})}{s_2 - s_1},$$

with

$$\sigma_i^{(k+1)} = \left[\frac{1}{n_i}\sum_{j=1}^{n_i} t_{ij}^{2\theta^{(k+1)}}\right]^{1/2\theta^{(k+1)}}, \quad i = 1, 2.$$

Step 3 Stop the iterative process when $|\theta^{(k+1)} - \theta^{(k)}| < \varepsilon, |\alpha_i^{(k+1)} - \alpha_i^{(k)}| < \varepsilon$ $(i = 0, 1)$, for some pre-assigned tolerance limit ε.

3.2 LSE

Let $T_{i1} < T_{i2} < \cdots < T_{in_i}$ $(i = 1, 2)$ be the random samples of the size n_i under stress s_i. From (2), one has

$$\ln[-\varPhi^{-1}[(1 - F(t_{ij}))/2] = \theta\ln t_{ij} - \beta_0 - \beta_1 s_i \quad (11)$$

where $\beta_0 = \theta\alpha_0$ and $\beta_1 = \theta\alpha_1$.

Let $\hat{F}_i$ be the empirical distribution function of $F(t)$ under s_i, where $F(t_{ij})$ equals j/n_i. In order to avoid $\ln 0$ in (11), we modify $\hat{F}_i$ to be $p_{ij} = j/(n_i+1)$ at t_{ij}. Therefore, the LSEs of β_0, β_1 and θ can be derived from

$$G(\alpha, \theta) = \sum_{i=1}^{2}\sum_{j=1}^{n_i} [y_{ij} - \theta\ln t_{ij} + \beta_0 + \beta_1 s_i]^2$$

where $y_{ij} = \ln[-\varPhi^{-1}[(1 - p_{ij})/2]$. Solving the normal equation

$$\frac{\partial G(\alpha,\theta)}{\partial\beta_0} = \frac{\partial G(\alpha,\theta)}{\partial\beta_1} = \frac{\partial G(\alpha,\theta)}{\partial\theta} = 0,$$

one has

$$\boldsymbol{A}\boldsymbol{\lambda} = \boldsymbol{B}$$

where $\boldsymbol{\lambda} = [\beta_0, \beta_1, \theta]'$

$$\boldsymbol{A} = \begin{bmatrix} \sum\limits_{i=1}^{2} n_i & \sum\limits_{i=1}^{2} n_i s_i & -\sum\limits_{i=1}^{2}\sum\limits_{j=1}^{n_i}\ln t_{ij} \\ \sum\limits_{i=1}^{2} n_i s_i & \sum\limits_{i=1}^{2} n_i s_i^2 & -\sum\limits_{i=1}^{2}\sum\limits_{j=1}^{n_i} s_i\ln t_{ij} \\ \sum\limits_{i=1}^{2}\sum\limits_{j=1}^{n_i}\ln t_{ij} & \sum\limits_{i=1}^{2}\sum\limits_{j=1}^{n_i} s_i\ln t_{ij} & -\sum\limits_{i=1}^{2}\sum\limits_{j=1}^{n_i}\ln^2 t_{ij} \end{bmatrix}$$

and

$$\boldsymbol{B} = \left[-\sum_{i=1}^{2}\sum_{j=1}^{n_i} y_{ij}, -\sum_{i=1}^{2}\sum_{j=1}^{n_i} s_i y_{ij}, -\sum_{i=1}^{2}\sum_{j=1}^{n_i} y_{ij}\ln t_{ij}\right]'.$$

Thus the LSEs of $\boldsymbol{\lambda}$, say $\tilde{\boldsymbol{\lambda}} = [\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\theta}]'$, can be derived as $\tilde{\boldsymbol{\lambda}} = \boldsymbol{A}^{-1}\boldsymbol{B}$, which implies that the LSEs of α_0, α_1 and θ are given by $\alpha_0^{LS} = \dfrac{\tilde{\beta}_0}{\tilde{\theta}}, \alpha_1^{LS} = \dfrac{\tilde{\beta}_1}{\tilde{\theta}}, \theta^{LS} = \tilde{\theta}$.

4. Confidence interval estimation

Confidence interval (CI) estimates are derived in this section including the asymptotic confidence intervals (ACIs) and the bootstrap confidence intervals (BCIs), respectively.

4.1 ACI

From (5), the second derivatives of $l(\alpha,\theta)$ with respect to α_0, α_1 and θ are

$$\frac{\partial^2 l(\alpha,\theta)}{\partial\alpha_0^2} = -2\theta\sum_{i=1}^{2}\sum_{j=1}^{n_i}\left(\frac{t_{ij}}{\sigma_i}\right)^{2\theta}$$

$$\frac{\partial^2 l(\alpha,\theta)}{\partial\alpha_0\partial\alpha_1} = -2\theta\sum_{i=1}^{2}\sum_{j=1}^{n_i}s_i\left(\frac{t_{ij}}{\sigma_i}\right)^{2\theta}$$

$$\frac{\partial^2 l(\alpha,\theta)}{\partial\alpha_0\partial\theta} = -\sum_{i=1}^{2}n_i + \sum_{i=1}^{2}\sum_{j=1}^{n_i}\left(\frac{t_{ij}}{\sigma_i}\right)^{2\theta} + 2\theta\sum_{i=1}^{2}\sum_{j=1}^{n_i}\left(\frac{t_{ij}}{\sigma_i}\right)^{2\theta}\ln t_{ij}$$

$$\frac{\partial^2 l(\alpha,\theta)}{\partial\alpha_1^2} = -2\theta^2\sum_{i=1}^{2}\sum_{j=1}^{n_i}s_i^2\left(\frac{t_{ij}}{\sigma_i}\right)^{2\theta}$$

$$\frac{\partial^2 l(\alpha,\theta)}{\partial\alpha_1\partial\theta} = -\sum_{i=1}^{2}n_i s_i + \sum_{i=1}^{2}\sum_{j=1}^{n_i}s_i\left(\frac{t_{ij}}{\sigma_i}\right)^{2\theta} + 2\theta\sum_{i=1}^{2}\sum_{j=1}^{n_i}s_i\left(\frac{t_{ij}}{\sigma_i}\right)^{2\theta}\ln t_{ij}$$

$$\frac{\partial^2 l(\alpha,\theta)}{\partial\theta^2} = -\sum_{i=1}^{2}\frac{n_i}{\theta^2} - 2\sum_{i=1}^{2}\sum_{j=1}^{n_i}\left(\frac{t_{ij}}{\sigma_i}\right)^{2\theta}\ln^2 t_{ij}.$$

From [10], one has

$$\int_0^\infty t^{v-1}\mathrm{e}^{-\vartheta t}\ln t\mathrm{d}t = \frac{\Gamma(v)}{\vartheta^v}[\psi(v)-\ln\vartheta], \quad v,\vartheta>0$$

$$\int_0^\infty t^{v-1}\mathrm{e}^{-\vartheta t}\ln^2 t\mathrm{d}t = \frac{\Gamma(v)}{\vartheta^v}\{[\psi(v)-\ln\vartheta]^2+\varsigma(2,v)\}$$

where $\Gamma(v)$ is the Gamma function, $\psi(v) = d\ln\Gamma(v)/dv$ and $\varsigma(z,v)$ is the Riemann's zeta function defined by

$$\varsigma(z,v) = \sum_{k=0}^{\infty}\frac{1}{(v+k)^z}, \quad z>1, v\neq 0,-1,\cdots$$

Denote $\eta = [\alpha_0,\alpha_1,\theta]$, the expected fisher information matrix $\boldsymbol{I}(\eta)$ is given by

$$\boldsymbol{I}(\eta) = [I_{ij}]_{3\times3} = \mathrm{E}[-\partial^2 l(\alpha,\theta)/\partial\eta_i\partial\eta_j]_{3\times3}$$

where $\eta_1=\alpha_0, \eta_2=\alpha_1, \eta_3=\theta$. Using previous results of [10], the items of expectations I_{ij} can be obtained as

$$I_{11} = \mathrm{E}\left[-\frac{\partial^2 l(\alpha,\theta)}{\partial\alpha_0^2}\right] = -2\theta\sum_{i=1}^{2}n_i$$

$$I_{12} = \mathrm{E}\left[-\frac{\partial^2 l(\alpha,\theta)}{\partial\alpha_0\partial\alpha_1}\right] = -2\theta\sum_{i=1}^{2}n_i s_i$$

$$I_{13} = \mathrm{E}\left[-\frac{\partial^2 l(\alpha,\theta)}{\partial\alpha_0\partial\theta}\right] = -\sum_{i=1}^{2}n_i([\psi(3/2)+\ln 2]+\ln\sigma_i)$$

$$I_{22} = \mathrm{E}\left[-\frac{\partial^2 l(\alpha,\theta)}{\partial\alpha_1^2}\right] = -2\theta^2\sum_{i=1}^{2}n_i s_i^2$$

$$I_{23} = \mathrm{E}\left[-\frac{\partial^2 l(\alpha,\theta)}{\partial\alpha_1\partial\theta}\right] = -\sum_{i=1}^{2}n_i s_i\{[\psi(3/2)+\ln 2]+\ln\sigma_i\}$$

$$I_{33} = \mathrm{E}\left[-\frac{\partial^2 l(\alpha,\theta)}{\partial\theta^2}\right] = \sum_{i=1}^{2}\frac{n_i}{\theta^2} + \sum_{i=1}^{2}\frac{n_i}{\theta^2}\{[\psi(3/2)+\ln 2]^2 + \varsigma(2,3/2)\} + 2\sum_{i=1}^{2}n_i\{\frac{\ln\sigma_i}{\theta}[\psi(3/2)+\ln 2]+\ln^2\sigma_i\}.$$

Under mild regularity conditions, the asymptotic distribution of the MLE $\hat\eta$ of η is

$$\hat\eta-\eta\to N(0,\boldsymbol{I}^{-1}(\eta))$$

where $\boldsymbol{I}^{-1}(\eta)$ is the inverse of the matrix $\boldsymbol{I}(\eta)$ with

$$\boldsymbol{I}^{-1}(\eta) = \begin{bmatrix} Var(\hat\alpha_0) & Cov(\hat\alpha_0,\hat\alpha_1) & Cov(\hat\alpha_0,\hat\theta) \\ Cov(\hat\alpha_0,\hat\alpha_1) & Var(\hat\alpha_1) & Cov(\hat\alpha_1,\hat\theta) \\ Cov(\hat\alpha_0,\hat\theta) & Cov(\hat\alpha_1,\hat\theta) & Var(\hat\theta)\end{bmatrix}.$$

Thus for $0<\gamma<1$, the $100(1-\gamma)\%$ ACI of η_k is given by $\hat\eta_k \pm z_{\gamma/2}\sqrt{Var(\hat\eta_k)}$ $(k=1,2,3)$, where z_γ is the upper γth quantile of the standard normal distribution.

4.2 BCI

MLE is a typical statistical method. However, in many practical situations since the sample size is not large, so the large-sample based inference like MLE-based asymptotic estimates may not be suitable and may even be misleading sometimes. In this subsection, parametric BCIs are constructed for unknown parameters, see [11] for details. The parametric bootstrap procedure is briefly described as follows:

Step 1 Compute MLE of η_k $(k = 1, 2, 3)$, say $\hat{\eta}_k$, based on data $t = \{t_{ij} : j = 1, \cdots, n_i, i = 1, 2\}$, where $\eta_1 = \alpha_0, \eta_2 = \alpha_1, \eta_3 = \theta$.

Step 2 Based on $\hat{\eta}_k$ $(k = 1, 2, 3)$, generate samples $t^* = \{t^*_{ij} : j = 1, \cdots, n_i, i = 1, 2\}$ with

$$t^*_{ij} = \hat{\sigma}_i[-\Phi^{-1}((1 - u_{ij})/\hat{\sigma}_i)]^{1/\hat{\theta}}$$

where for $i = 1, 2, u_{ij}$ is generated independent observations from the uniform distribution $U(0, 1)$ of sample size n_i.

Step 3 Compute MLE of η_k $(k = 1, 2, 3)$, say $\hat{\eta}^*_k$, based on t^*.

Step 4 Repeat Steps 2 and 3 N times and generate N bootstrap estimate of η_k $(k = 1, 2, 3)$.

Step 5 Arrange all $\hat{\eta}^*_k$ $(k = 1, 2, 3)$ in an ascending order, one can obtain the bootstrap sample for η_k as $\eta_k^{*[1]}, \cdots, \eta_k^{*[N]}$.

Let $G(z) = P(\eta_k < z)$ be the CDF of η_k $(k = 1, 2, 3)$. Define $\eta_{k\text{boot}} = G^{-1}(z)$ for given z. For $0 < \gamma < 1$, the bootstrap $100(1 - \gamma)\%$ CI of η_k is $(\hat{\eta}_{k\text{boot}}(\gamma/2), \hat{\eta}_{k\text{boot}}((1 - \gamma)/2)), \quad k = 1, 2, 3$.

5. Numerical illustration

5.1 Simulation studies

In this subsection, simulation is conducted to investigate the performance of point estimates in terms of average bias (AB) and mean square error (MSE), and interval estimates in terms of coverage probability (CP) and average width (AW), respectively. For different choices of $n_1, n_2, \alpha_0, \alpha_1, \theta$ values, previous four criteria quantities are obtained based on 10 000 replications and the results are presented in Table 1 to Table 4 with $\gamma = 0.10$.

Table 1 ABs and MSEs (within bracket) for parameters at $\boldsymbol{a} = (0.3, -0.6), \theta = 1, (s_1, s_2) = (0.3, 0.8)$

(n_1, n_2)	a_0		a_1		θ	
	MLE	LSE	MLE	LSE	MLE	LSE
(10,15)	0.103 8[0.012 6]	0.301 7[0.127 6]	0.087 9[0.012 9]	0.387 6[0.169 2]	0.062 4[0.009 1]	0.265 4[0.096 9]
(10,20)	0.095 1[0.011 9]	0.254 8[0.086 2]	0.063 4[0.007 5]	0.354 8[0.138 5]	0.056 8[0.005 3]	0.237 9[0.069 2]
(15,20)	0.084 6[0.009 4]	0.196 4[0.055 6]	0.052 0[0.005 6]	0.289 7[0.103 1]	0.042 3[0.003 2]	0.184 7[0.042 6]
(25,35)	0.053 1[0.005 7]	0.126 5[0.021 7]	0.039 1[0.002 8]	0.204 3[0.053 9]	0.021 0[0.000 9]	0.112 5[0.020 6]
(45,50)	0.037 9[0.003 1]	0.099 2[0.012 1]	0.026 7[0.001 7]	0.123 8[0.021 4]	0.019 7[0.000 3]	0.084 9[0.017 2]

Table 2 CPs and AWs (within bracket) for parameters at $\boldsymbol{a} = (0.3, -0.6), \theta = 1, (s_1, s_2) = (0.3, 0.8)$

(n_1, n_2)	ACI			BCI		
	a_0	a_1	θ	a_0	a_1	θ
(10,15)	0.883 6[0.311 7]	0.872 1[0.920 3]	0.884 5[0.265 4]	0.882 7[0.281 2]	0.871 2[0.754 1]	0.884 2[0.221 4]
(10,20)	0.885 4[0.276 8]	0.878 3[0.891 8]	0.887 3[0.248 3]	0.884 6[0.264 3]	0.874 9[0.687 9]	0.886 7[0.198 9]
(15,20)	0.889 1[0.254 9]	0.883 9[0.763 2]	0.891 6[0.186 7]	0.887 9[0.225 9]	0.882 5[0.569 3]	0.891 0[0.164 5]
(25,35)	0.896 3[0.173 5]	0.895 6[0.542 0]	0.897 8[0.135 2]	0.891 4[0.164 0]	0.893 1[0.388 6]	0.895 8[0.111 1]
(45,50)	0.902 1[0.149 2]	0.900 9[0.464 7]	0.910 2[0.121 0]	0.896 2[0.125 8]	0.898 9[0.341 1]	0.901 3[0.087 9]

Table 3 ABs and MSEs (within bracket) for parameters at $\boldsymbol{a} = (0.1, -0.9), \theta = 1.2, (s_1, s_2) = (0.5, 1.1)$

(n_1, n_2)	a_0		a_1		θ	
	MLE	LSE	MLE	LSE	MLE	LSE
(15,10)	0.215 3[0.053 4]	0.378 9[0.175 7]	0.134 6[0.026 0]	0.432 7[0.231 4]	0.119 2[0.047 7]	0.321 6[0.145 3]
(20,10)	0.196 4[0.040 5]	0.325 1[0.121 9]	0.123 1[0.021 4]	0.398 9[0.168 6]	0.108 5[0.041 2]	0.312 4[0.107 7]
(20,15)	0.167 1[0.036 5]	0.289 7[0.084 8]	0.104 2[0.013 1]	0.351 8[0.130 5]	0.093 6[0.014 0]	0.235 9[0.079 1]
(35,25)	0.113 0[0.014 7]	0.155 4[0.036 6]	0.086 7[0.011 3]	0.269 4[0.086 0]	0.064 8[0.007 5]	0.111 0[0.016 4]
(50,45)	0.064 8[0.008 3]	0.097 9[0.010 2]	0.052 3[0.007 3]	0.121 2[0.033 6]	0.036 4[0.002 8]	0.072 1[0.006 8]

Table 4 CPs and AWs (within bracket) for parameters at $\boldsymbol{a} = (0.1, -0.9), \theta = 1.2, (s_1, s_2) = (0.5, 1.1)$

(n_1, n_2)	ACI			BCI		
	a_0	a_1	θ	a_0	a_1	θ
(15,10)	0.887 3[0.452 7]	0.884 5[0.892 4]	0.890 1[0.319 2]	0.885 4[0.432 0]	0.883 5[0.736 2]	0.886 2[0.291 4]
(20,10)	0.891 1[0.423 2]	0.889 4[0.761 8]	0.892 3[0.283 4]	0.887 9[0.387 9]	0.886 4[0.694 1]	0.891 3[0.256 3]
(20,15)	0.893 2[0.345 1]	0.890 6[0.694 6]	0.894 1[0.231 5]	0.891 2[0.302 1]	0.891 0[0.634 8]	0.893 5[0.201 0]
(35,25)	0.901 0[0.231 7]	0.895 4[0.472 9]	0.900 2[0.183 1]	0.896 8[0.198 7]	0.892 7[0.451 7]	0.897 4[0.164 5]
(50,45)	0.910 3[0.184 3]	0.904 6[0.323 6]	0.906 3[0.149 8]	0.908 1[0.153 4]	0.898 6[0.298 0]	0.903 2[0.127 3]

From Table 1 to Table 4, it is observed that when n_1 or n_2 or both (n_1, n_2) increases, the ABs and MSEs decrease for all MLEs and LSEs of model parameters. For the fixed (n_1, n_2), ABs and MSEs of $\alpha_0, \alpha_1, \theta$ are relatively smaller than those of LSEs, which implies that the MLEs of parameters $\alpha_0, \alpha_1, \theta$ have better performances than the LSEs. Furthermore, it is also noted that, AWs of BCIs are usually shorter than ACIs under fix sample sizes, the ACIs, however, are slight closer CPs to the nominal level comparing to BCIs. To sum up, although the LSEs of $\alpha_0, \alpha_1, \theta$ have concise expressions, we would recommend to use the MLEs for point estimates, and for interval estimates, if one wants CIs with a shorter interval length, BCIs may be an appropriate choice; otherwise if one wants CIs having closer CPs to the nominal level and the interval lengths are not a major concern, the ACIs can be employed which provide a balance between coverage probability and average width.

5.2 Data analysis

In this example, the following simulated simple constant-stress ALT random samples are generated from (2) when $\alpha_0 = 0.2$, $\alpha_1 = -0.5$, $\theta = 1$, $s_1 = 0.25$, $s_2 = 0.75$ as follows:

$$
\begin{aligned}
s_1 :\ & 0.049\,9 \quad 0.096\,2 \quad 0.271\,7 \quad 0.371\,5 \quad 0.598\,8 \\
& 0.649\,5 \quad 0.691\,6 \quad 0.746\,0 \quad 0.746\,5 \quad 0.798\,4 \\
& 1.207\,8 \quad 1.273\,8 \quad 1.289\,4 \quad 1.853\,5 \quad 2.350\,3 \\
s_2 :\ & 0.012\,6 \quad 0.091\,4 \quad 0.121\,2 \quad 0.216\,8 \quad 0.299\,5 \\
& 0.346\,6 \quad 0.347\,2 \quad 0.411\,4 \quad 0.713\,3 \quad 0.756\,6 \\
& 0.808\,7 \quad 0.941\,9 \quad 1.249\,7 \quad 1.386\,1 \quad 1.744\,1.
\end{aligned}
$$

Both point and interval estimates are derived in Table 5, where CIs are obtained with $\gamma = 0.10$. Meanwhile, in order to show that the profile log-likelihood function $l(\theta)$ is unimodal and has a unique solution, the curve of $l(\theta)$ is plotted in Fig. 1.

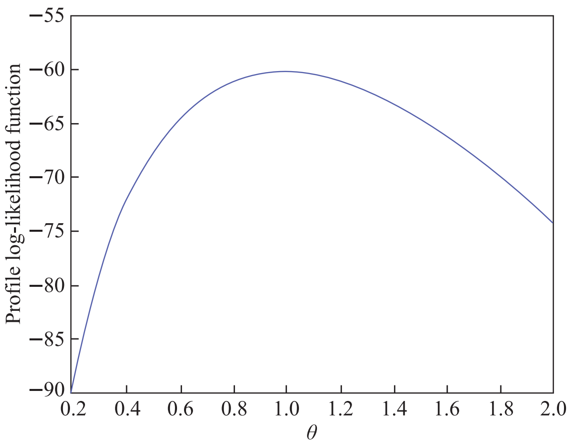

Fig. 1 Profile log-likelihood function of θ

Table 5 Point estimates and bias (within bracket) and interval estimates and interval lengths (within bracket) under simulated data

Estimate	α_0	α_1	θ
MLE	0.196 1[0.003 9]	–0.550 5[0.050 5]	0.995 0[0.005 0]
LSE	0.499 6[0.299 6]	–0.906 8[0.406 8]	0.721 9[0.278 1]
ACI	(0.047 9,0.344 3) [0.296 4]	(–0.996 9,–0.104 0) [0.892 9]	(0.950 4,1.139 6) [0.189 2]
BCI	(0.113 2,0.291 0) [0.177 8]	(–0.835 4,–0.271 8) [0.563 6]	(0.972 6,1.115 7) [0.143 1]

From Table 5, it is seen that MLEs have better performances than LSEs where the MLEs feature smaller bias than LSEs, and that the interval lengths of BCIs are shorter than ACIs, which also shows BCIs are superior to ACIs in terms of interval lengths.

6. Conclusions

In this paper, the estimation is discussed for the constant-stress ALT when the lifetime of products follows the GHN distribution. The point estimates of parameters are derived by using methods of the maximum likelihood and least-square estimations, and the interval estimates are constructed through asymptotic theory and bootstrap procedure. Although the paper considers inference work under the simple constant-stress model, the results can be extended to k stages $(k \geqslant 3)$ constant-stress ALT. In addition, it is worth mentioning that although the life-stress model (4) is log-linear, it can be also extended to general cases. For instance, the life-stress model can be assumed as $\ln \sigma_i = \alpha_0 + \alpha_1 \varphi(s_i)$. According to [8], the function φ follows expressions in practice such as $\varphi = 1/s$ in the Arrhenius model; $\varphi = \ln s$ in the inverse power model; and $\varphi = s$ in the exponential model. The results proposed in this paper can be derived directly under this assumption.

In future work, reliability inference such as testing design and sampling plan of the ALT under complete and censored data for the GHN model seems also interesting which needs further study.

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