Consensus of second-order nonlinear multi-agent systems via sliding mode observer and controller

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Abstract: This paper investigates the consensus problem of second-order nonlinear multi-agent systems (MASs) via the sliding mode control (SMC) approach. The velocity of each agent is assumed to be unmeasurable. A second-order sliding mode observer is designed to estimate the velocity. Then a distributed discontinuous control law based on first-order SMC is presented to solve the consensus problem. Moreover, to overcome the chattering problem, two controllers based on the boundary layer method and the super-twisting algorithm respectively are presented. It is shown that the MASs will achieve consensus under some given conditions. Some examples are provided to demonstrate the effectiveness of the proposed control laws.

Keywords: nonlinear multi-agent system, sliding mode observer, consensus, sliding mode controller.

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1. Introduction

Distributed cooperative control of multi-agent systems (MASs), such as formation [1,2], flocking [3,4], synchronization of coupled oscillators [5,6], has drawn much attention as a result of their scalability and robustness in node failures. As an important fundamental problem of above, consensus with its aim at making the agents swarm achieve an identical state has been extensively studied.

Recently, many results have been made on consensus control for both first-order and second-order MASs [7–9]. In [7], Yu et al. investigated second-order consensus in MASs with sampling both current and some past position data. Zhu et al. considered the robust second-order consensus tracking problem for the MASs with disturbances and unmodeled agent dynamics [8]. Lin and Jia investigated the second-order consensus problem of discrete-time

MASs with nonuniform time-delays and switching topologies [9]. Furthermore, some other control strategies have been developed for consensus of MASs, for instance H_∞ control [10,11], sampled-data control [12], pinning control [13,14], sliding mode control (SMC) [15,16] and robust control [8,17–19]. Some constraints including time delay [12], data packet dropout [12] and parameter uncertainties [11,17] have also been considered. Those aforementioned studies are mostly considered the consensus problem of linear MASs. However, there are still many challenges for the nonlinear MASs consensus control problem, especially with unmeasurable states.

Unmeasurable states are usually observed by designing linear observers in existing researches. For example, in [20], a novel consensus protocol based on the state observer with a fast convergence speed was presented for linear MASs, where the unmeasurable states are observed by the well designed linear observers. In [21], Hu et al. considered the distributed pinning consensus control problem for the nonlinear MASs with the local line observer and distributed pinning observer. It is noted that, the linear approaches cannot guarantee the observer and controller converge in finite time for the nonlinear system with uncertainties.

SMC has been widely used to design observers and consensus protocols for MASs [22–24] as a result of its advantages, for instance, high reliability, strong robustness, fast system response and so on. In [22], the SMC is applied for first-order linear MASs with disturbances. In [23–26], the terminal SMC (TSMC) and twisting second-order SMC are applied for the consensus of second-order MASs with disturbances. In [27], an integral sliding mode approach was used to solve the finite-time consensus problem of second-order MASs with bounded disturbances, however, the chattering problem is ignored.

Motivated by these discussions, we address the second-

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order consensus problem of a class of nonlinear MASs subject to unmeasurable state via the sliding mode control approach. The main contributions of this paper are listed as follows. Firstly, compared with [27], the nonlinear dynamic in MASs is only restricted to be uniformly bounded without needing to know its derivative information, and a finite-time observer and a first-order sliding mode controller are presented for the consensus of second-order nonlinear MASs with unmeasurable velocity. Secondly, due to the chattering problem in the first-order sliding mode controller, modified consensus controllers based on the boundary layer method and the super-twisting algorithm are presented respectively. In conclusion, this paper investigates the consensus problem of second-order nonlinear MASs under a relaxed assumption of nonlinear dynamic via the finite-time observer and the sliding mode controller.

This paper is organized as follows. In Section 2, the graph theory and the second-order nonlinear MASs are introduced. Distributed observer and controller based on SMC for the consensus of second-order nonlinear multi-agent systems are presented in Section 3. Section 4 investigates the modified SMC based on the boundary layer method and the super-twisting algorithm. In Section 5, simulation results are provided and Section 6 concludes this paper.

2. Problem formulation and preliminaries

2.1 Graph theory

A graph $G = (V, E)$ is used to describe the topology of MASs, $V = \{1, 2, \dots, n\}$ denotes the vertex set and $E \subseteq \{(i, j) | i, j \in V, i \neq j\}$ denotes the edge set, respectively. The topology is considered to be directed in this paper. If $(i, j) \in E$, we say the agent j can receive information from its neighbor agent i . If node $j \in N_i(t) = \{j | (j, i) \in E, j \neq i\}$, we say the node j is a neighbour of the agent i . If there is a node in graph G which can reach all other nodes, then we say G contains a spanning tree. Define the Laplacian matrix $L(G)$ (simply L) as $L = D - A$, where $A = [a_{ij}]$ is the adjacency matrix with $a_{ij} = 1$, $\forall (i, j) \in E$ and $a_{ij} = 0$ otherwise and $D = \text{diag}(\sum_{j \neq i} a_{ij})$ is the degree matrix. Other relevant graph theory knowledge can be found in [28].

Lemma 1 [29] For $x_i \in \mathbf{R}$ ($i = 1, 2, \dots, n$), $\alpha \in (0, 1]$, then $(\sum_{i=1}^n |x_i|)^\alpha \leq \sum_{i=1}^n |x_i|^\alpha$.

2.2 Problem formulation

The second-order nonlinear MAS dynamics are described

as follows:

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = f_i(t, x_i(t), v_i(t)) + u_i(t) \end{cases}, i = 1, \dots, N \quad (1)$$

where $x_i(t), v_i(t) \in \mathbf{R}$ are the position and velocity, respectively. $f_i : \mathbf{R} \mapsto \mathbf{R}$ is a continuously differentiable function, $u_i(t) \in \mathbf{R}$ is the control input. Although the system is considered to be second-order, the proposed method in this paper can be extended to the high-order case. The following assumption is needed in this paper.

Assumption 1 $f_i(t, x_i(t), v_i(t))$ is uniformly bounded, i.e., there exists a known $\eta > 0$ satisfying

$$|f_i(t, x_i(t), v_i(t))| \leq \eta. \quad (2)$$

Lemma 2 [30] Consider the following first-order MASs

$$\dot{x}_i = \sum_{j \in N_i} a_{ij}(x_j - x_i),$$

then the MASs can achieve consensus, if and only if the system topology contains a spanning tree.

In this paper, the consensus is said to be achieved if and only if the following conditions hold:

$$\begin{cases} \lim_{t \rightarrow \infty} \|x_i - x_j\| = 0 \\ \lim_{t \rightarrow \infty} \|v_i - v_j\| = 0 \end{cases}, i, j \in V. \quad (3)$$

Remark 1 In this paper, the MASs finally achieve average consensus which means the average value of the initial states, of course it is static consensus because the velocities of agents decay to zero finally.

3. Main result

In this section, we first present the second-order sliding mode observer based on the super-twisting algorithm [31] for MASs, and then according to the separation principle [32] and proposed interacted sliding mode surfaces, three distributed sliding mode consensus controllers are designed to eliminate the chatting.

3.1 Second-order sliding mode observer design

Since the velocities of agents are assumed to be unmeasurable, we need to estimate the velocity firstly. The super-twisting second-order sliding mode observer as follows by developing the design of the super-twisting sliding mode observer is given as follows:

$$\begin{cases} \dot{\hat{x}}_i = \hat{v}_i + w_1 \\ \dot{\hat{v}}_i = f_i(t, \hat{x}_i, \hat{v}_i) + u_i + w_2 \end{cases}, i = 1, \dots, N \quad (4)$$

where $\hat{x}_i, \hat{v}_i$ are the estimation variables of x_i, v_i , and w_1, w_2 are auxiliary variables in the following form:

$$\begin{cases} w_1 = \alpha_1 |x_i - \hat{x}_i|^{1/2} \text{sgn}(x_i - \hat{x}_i) \\ w_2 = \alpha_2 \text{sgn}(x_i - \hat{x}_i) \end{cases} \quad (5)$$

where $\alpha_1, \alpha_2 > 0$ are to be determined, and $\text{sgn}(\cdot)$ is the following sign function

$$\text{sgn}(\theta_i) = \begin{cases} 1, & \theta_i > 0 \\ 0, & \theta_i = 0 \\ -1, & \theta_i < 0 \end{cases}$$

Define $\tilde{x}_i = x_i - \hat{x}_i, \tilde{v}_i = v_i - \hat{v}_i$, and then according to (4) and (5), one can get the error dynamics

$$\begin{cases} \dot{\tilde{x}}_i = \tilde{v}_i - \alpha_1 |\tilde{x}_i|^{1/2} \text{sgn}(\tilde{x}_i) \\ \dot{\tilde{v}}_i = F_i(t, x_i, v_i, \hat{x}_i, \hat{v}_i) - \alpha_2 \text{sgn}(\tilde{x}_i) \end{cases} \quad (6)$$

where $F_i(t, x_i, v_i, \hat{x}_i, \hat{v}_i) = f_i(t, x_i, v_i) - f_i(t, \hat{x}_i, \hat{v}_i)$, and $|F_i(t, x_i, v_i, \hat{x}_i, \hat{v}_i)| \leq 2\eta$.

Now we give the following lemma to verify the effectiveness of the proposed observer.

Lemma 3 Suppose that Assumption 1 holds. For the system (1) and the observer (4), if α_1, α_2 satisfy

$$\alpha_1 > 2, \quad \alpha_2 > \frac{\alpha_1^3 + 4(4\alpha_1 - 8)\eta^2}{\alpha_1(4\alpha_1 - 8)} \quad (7)$$

then the system (6) can converge to the origin in finite time.

Proof Consider the following Lyapunov function

$$V_i(\tilde{x}_i, \tilde{v}_i) = 2\alpha_2 |\tilde{x}_i| + \frac{1}{2}(\tilde{v}_i^T \tilde{v}_i + \theta_i^T \theta_i) = \xi_i^T P \xi_i \quad (8)$$

where $\theta_i^T = -\tilde{v}_i + \alpha_1 |\tilde{x}_i|^{1/2} \text{sgn}(\tilde{x}_i)$, $\xi_i^T = [|\tilde{x}_i|^{1/2} \text{sgn}(\tilde{x}_i) \quad \tilde{v}_i]$, $P = \frac{1}{2} \begin{bmatrix} \alpha_1^2 + 4\alpha_2 & -\alpha_1 \\ -\alpha_1 & 2 \end{bmatrix}$. Note that $V_i(\tilde{x}_i, \tilde{v}_i)$ is continuous differentiable except $\tilde{x}_i = 0$. It is positively definite and radially unbounded if $\alpha_2 > 0$, i.e.

$$\lambda_{\min}\{P\} \|\xi_i\|^2 \leq V_i(\tilde{x}_i, \tilde{v}_i) \leq \lambda_{\max}\{P\} \|\xi_i\|^2 \quad (9)$$

where $\lambda_{\min}\{\cdot\}$ and $\lambda_{\max}\{\cdot\}$ are minimum and maximum eigenvalues of the matrix, respectively. $\|\xi_i\|^2 = |\tilde{x}_i| + \tilde{v}_i^2$ is Euclidean norm of ξ_i . Furthermore,

$$|\tilde{x}_i|^{1/2} \leq \|\xi_i\| \leq \frac{V_i(\tilde{x}_i, \tilde{v}_i)^{1/2}}{\lambda_{\min}^{1/2}\{P\}} \quad (10)$$

The derivative of ξ_i is

$$\dot{\xi}_i = |\tilde{x}_i|^{-1/2} (A\xi_i + B\tilde{F}_i) \quad (11)$$

where $A = \begin{bmatrix} -\frac{\alpha_1}{2} & \frac{1}{2} \\ -\alpha_2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\tilde{F}_i = |\tilde{x}_i|^{1/2} F(t, x_i, v_i, \hat{x}_i, \hat{v}_i)$.

One can get the derivative of $V_i(\tilde{x}_i, \tilde{v}_i)$ along (6) as

$$\dot{V}_i(\tilde{x}_i, \tilde{v}_i) = |\tilde{x}_i|^{-1/2} \begin{bmatrix} \xi_i \\ \tilde{F}_i \end{bmatrix}^T$$

$$\begin{aligned} & \begin{bmatrix} A^T P + P A & P B \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} \xi_i \\ \tilde{F}_i \end{bmatrix} \leq \\ & |\tilde{x}_i|^{-1/2} \left(\begin{bmatrix} \xi_i \\ \tilde{F}_i \end{bmatrix}^T \cdot \begin{bmatrix} A^T P + P A & P B \\ B^T P & 0 \end{bmatrix} \cdot \begin{bmatrix} \xi_i \\ \tilde{F}_i \end{bmatrix} + \right. \\ & \left. (2\eta)^2 |\tilde{x}_i| - \tilde{F}_i^2 \right) \leq |\tilde{x}_i|^{-1/2} \xi_i^T (A^T P + P A + \\ & (2\eta)^2 C^T C + P B B^T P) \xi_i = -|\tilde{x}_i|^{-1/2} \xi_i^T Q \xi_i \quad (12) \end{aligned}$$

where $C = [1 \quad 0]$, $-Q = A^T P + P A + (2\eta)^2 C^T C + P B B^T P$, and

$$Q = \begin{bmatrix} \varpi & \frac{\alpha_1}{2} - \frac{\alpha_1^2}{2} \\ \frac{\alpha_1}{2} - \frac{\alpha_1^2}{2} & \frac{\alpha_1}{2} - 1 \end{bmatrix},$$

where $\varpi = \frac{\alpha_1^3}{2} - \frac{\alpha_1^2}{4} + \alpha_1 \alpha_2 - 4\eta^2$. Define α_1, α_2 as in (7), and then Q is a positive definite matrix. Thus one has

$$\begin{aligned} \dot{V}_i(\tilde{x}_i, \tilde{v}_i) & \leq -|\tilde{x}_i|^{-1/2} \xi_i^T Q \xi_i \leq \\ & -|\tilde{x}_i|^{-1/2} \lambda_{\min}\{Q\} \|\xi_i\|^2 \leq -\lambda_{\min}\{Q\} \|\xi_i\|. \quad (13) \end{aligned}$$

According to (10) and (13), one has

$$\dot{V}_i(\tilde{x}_i, \tilde{v}_i) \leq -\gamma V_i(\tilde{x}_i, \tilde{v}_i)^{1/2} \quad (14)$$

where $\gamma = \frac{\lambda_{\min}\{Q\}}{\lambda_{\max}\{P\}}$. Following from the comparison

principle [33], when $t \geq \max_{1 \leq i \leq N} \frac{2V_i(\tilde{x}_i(0), \tilde{v}_i(0))^{1/2}}{\gamma}$, we have $V_i(\tilde{x}_i, \tilde{v}_i) = 0$. The proof is thus completed. $\square$

Remark 2 The finite convergence time is apparently related to the matrix Q and P . More directly, it is related to the parameters α_1, α_2 and bound of the nonlinear dynamic η . In fact, the adaptive sliding mode observer can be further designed to optimize the convergence time according to the adaptive control schemes proposed in [33,34].

3.2 First-order sliding mode controller design

In this subsection, the basic idea is to design a sliding mode surface for each agent such that it will converge to the sliding mode surface. Then, if the sliding mode surfaces can converge to one common point, then the consensus is achieved. The first-order sliding mode controller is designed as follows:

$$u_i = \sum_{j \in N_i} a_{ij}(\hat{v}_j - \hat{v}_i) - (\beta + \eta) \text{sgn}(s_i) \quad (15)$$

where β is a positive constant, and the switching function is $s_i = \hat{v}_i - k \sum_{j \in N_i} a_{ij}(\hat{x}_j - \hat{x}_i)$, in which k is a positive constant called consensus gain, we make $k = 1$ in this paper.

Then we can obtain the following results.

Theorem 1 Suppose that Assumption 1 holds, and the communication topology contains a spanning tree. Under the first-order sliding mode controller (15), the MAS (1) will achieve consensus asymptotically.

Proof We firstly prove that the system state can reach the sliding mode surface $s_i = 0$ in finite time. Choosing the Lyapunov function as $V_i(s_i) = \frac{1}{2}s_i^T s_i$ and taking the derivative of V_i as

$$\begin{aligned}\dot{V}_i(s_i) &= s_i \dot{s}_i = s_i [\dot{\hat{v}}_i - \sum_{j \in N_i} a_{ij}(\hat{v}_j - \hat{v}_i)] = \\ &= s_i [\dot{v}_i - \sum_{j \in N_i} a_{ij}(v_j - v_i)] = \\ &= s_i [f_i(t, x_i, v_i) + u_i - \sum_{j \in N_i} a_{ij}(v_j - v_i)] = \\ &= s_i [f_i(t, x_i, v_i) - (\beta + \eta) \text{sgn}(s_i)] \leq \\ &= |s_i| \eta - (\beta + \eta) |s_i| = -\beta |s_i|,\end{aligned}\quad (16)$$

one can obtain

$$\dot{V}_i(s_i) \leq -\sqrt{2}\beta V_i(s_i)^{1/2} < 0.$$

According to the comparison principle [35], when $t \geq \frac{\max_{1 \leq i \leq N} \sqrt{2V_i(s_i(0))}}{\gamma}$, one has $s_i = v_i - \sum_{j \in N_i} a_{ij}(x_j - x_i) = 0$, i.e., v_i ($i = 1, \dots, N$) will converge to their sliding mode surfaces in finite time. Then the MAS (1) can be written as

$$\dot{x}_i = v_i = \sum_{j \in N_i} a_{ij}(x_j - x_i). \quad (17)$$

According to Lemma 2, the MAS (1) will achieve consensus, which completes the proof. $\square$

Remark 3 According to Lemma 3, the designed second-order sliding mode observer error can converge to the zero within finite time, according to separation principle [32], the design of the observer and controller can be considered and analysed separately, which means the observer states in controller can be replaced by real states in (16).

3.3 Sliding mode controller based on boundary layer

In SMC, the chattering problem is one of the important reasons to restrict its application. It is noted that the first-order sliding mode controller proposed in Section 3.2 will generate serious chattering. In order to overcome the chattering problem, Slotine et al. [36] introduced the definition of the boundary layer and the quasi-sliding mode into SMC. The quasi-sliding mode means system trajectory is limited into

a Δ neighbourhood of the ideal sliding mode, where Δ is called as the boundary layer of the sliding mode.

In this part, we replace the $\text{sgn}(s_i)$ by the saturation function $\text{sat}(s_i)$ to achieve quasi-SMC as follows:

$$u_i = \sum_{j \in N_i} a_{ij}(\hat{v}_j - \hat{v}_i) - (\beta + \eta) \text{sat}(s_i) \quad (18)$$

where the switching function $s_i = v_i - \sum_{j \in N_i} a_{ij}(x_j - x_i)$

with

$$\text{sat}(s_i) = \begin{cases} \text{sgn}(s_i), & |s_i| > \Delta \\ s_i/\Delta, & |s_i| \leq \Delta \end{cases}.$$

$\Delta > 0$ is the thickness of the boundary layer.

Theorem 2 Suppose that Assumption 1 holds, and the communication topology contains a spanning tree. Under the sliding mode controller based on boundary layer in (18), the MAS (1) will achieve consensus asymptotically.

Proof The proof of the case $|s_i| > \Delta$ is the same as the first-order sliding mode controller design, and we just prove the case $|s_i| \leq \Delta$. Choose the Lyapunov function $V_i(s_i) = \frac{1}{2}s_i^T s_i$ and take the derivative of V_i as

$$\begin{aligned}\dot{V}_i(s_i) &= s_i \dot{s}_i = \\ &= s_i [f_i(t, x_i, v_i) - (\beta + \eta) \text{sat}(s_i)] = \\ &= s_i [f_i(t, x_i, v_i) - (\beta + \eta) \frac{s_i}{\Delta}] \leq \\ &= |s_i| |f_i(t, x_i, v_i)| - (\beta + \eta) \frac{s_i^2}{\Delta} \leq \\ &= \Delta \eta - (\beta + \eta) \Delta = -\beta \Delta < 0.\end{aligned}\quad (19)$$

As demonstrated in Theorem 1, the MAS (1) will achieve consensus under the controller (18), which completes the proof. $\square$

Remark 4 Quasi-SMC can only drive the agent state converge to the neighborhood of sliding surface. Theoretically, the boundary layer thickness Δ can be designed arbitrarily small to be close to the sliding mode surface, but still cannot guarantee the state converge to the sliding surface. In the worst case, the system state can only reach the boundary of the boundary layer, impacting on the steady state of the system.

3.4 Second-order sliding mode controller design

We design the super-twisting second-order sliding mode controller. Compared with the above two controllers, the super-twisting second-order sliding mode controller can keep the invariability of the first-order sliding mode, overcome the chattering problem and improve the system performance.

The super-twisting second-order sliding mode controller is proposed as follows:

$$\begin{cases} u_i = \sum_{j \in N_i} a_{ij}(\hat{v}_j - \hat{v}_i) - \\ \beta_1 |s_i|^{1/2} \text{sgn}(s_i) + z_i \\ \dot{z}_i = -\beta_2 \text{sgn}(s_i) \end{cases} \quad (20)$$

where the switching function $s_i = v_i - \sum_{j \in N_i} a_{ij}(x_j - x_i)$, and β_1, β_2 are determined constants.

The derivation of s_i is given as follows:

$$\begin{cases} \dot{s}_i = f_i(t, x_i, v_i) - \beta_1 |s_i|^{1/2} \text{sgn}(s_i) + z_i \\ \dot{z}_i = -\beta_2 \text{sgn}(s_i) \end{cases} \quad (21)$$

According to Assumption 1 and $s_i = v_i - \sum_{j \in N_i} a_{ij}(x_j - x_i)$, we can rewrite $f_i(t, x_i, v_i)$ as $f(t, s_i)$. The term $f(t, s_i)$ is uniformly bounded, i.e., there exists a $\varsigma > 0$ satisfying $|f_i(t, s_i)| \leq \varsigma$. Then (21) can be written as

$$\begin{cases} \dot{s}_i = f_i(t, s_i) - \beta_1 |s_i|^{1/2} \text{sgn}(s_i) + z_i \\ \dot{z}_i = -\beta_2 \text{sgn}(s_i) \end{cases} \quad (22)$$

According to the controller in (20) and (22), we have the following results.

Theorem 3 Suppose that Assumption 1 holds. For the MAS (1) and the controller (20), if β_1 and β_2 satisfy

$$\beta_1 > 2, \quad \beta_2 > \frac{\frac{3}{4}\beta_1^3 - 2\beta_1^2 + (\beta_1 - 2)\varsigma^2}{(\beta_1 + 1)^2 - 9}, \quad (23)$$

then, the system (22) converges to the origin in finite-time.

Proof Consider the following Lyapunov function

$$V = \sum V_i(s_i, z_i) =$$

$$\sum (2\beta_2 |s_i| + \frac{1}{2}(z_i^T z_i + v_i^T v_i)) = \sum \zeta_i^T M \zeta_i \quad (24)$$

where $v_i^T = \beta_1 |s_i|^{1/2} \text{sgn}(s_i) - z_i$, $\zeta_i^T = [|s_i|^{1/2} \text{sgn}(s_i) \quad z_i]$, $M = \frac{1}{2} \begin{bmatrix} \beta_1^2 + 4\beta_2 & -\beta_1 \\ -\beta_1 & 2 \end{bmatrix}$.

The derivative of ζ_i is

$$\dot{\zeta}_i = |\dot{s}_i|^{-1/2} (\tilde{A} \zeta_i + \tilde{B} \hat{F}_i) \quad (25)$$

where $\tilde{A} = \begin{bmatrix} -\frac{\beta_1}{2} & \frac{1}{2} \\ -\beta_2 & 0 \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\hat{F}_i = |s_i|^{1/2} f(t, s_i)$.

One can get the derivative of $V_i(s_i, z_i)$ along (6) as

$$\dot{V} = \sum \dot{V}_i(s_i, z_i) = \sum \left(|s_i|^{-1/2} \begin{bmatrix} \zeta_i \\ \hat{F}_i \end{bmatrix}^T \right.$$

$$\left. \begin{bmatrix} \tilde{A}^T M + M \tilde{A} & M \tilde{B} \\ \tilde{B}^T M & 0 \end{bmatrix} \cdot \begin{bmatrix} \zeta_i \\ \hat{F}_i \end{bmatrix} \right) \leq$$

$$\sum \left(|s_i|^{-1/2} \left(\begin{bmatrix} \zeta_i \\ \hat{F}_i \end{bmatrix}^T \right.$$

$$\left. \begin{bmatrix} \tilde{A}^T M + M \tilde{A} & M \tilde{B} \\ \tilde{B}^T M & 0 \end{bmatrix} \cdot \begin{bmatrix} \zeta_i \\ \hat{F}_i \end{bmatrix} + \varsigma^2 |s_i| - \hat{F}_i^2 \right) \leq$$

$$\begin{aligned} & \sum (|s_i|^{-1/2} \zeta_i^T (\tilde{A}^T M + M \tilde{A} + \\ & \varsigma^2 \tilde{C}^T \tilde{C} + M \tilde{B} \tilde{B}^T M) \zeta_i) = \\ & \sum (-|s_i|^{-1/2} \zeta_i^T \tilde{Q} \zeta_i) \end{aligned} \quad (26)$$

where $\tilde{C} = [1 \quad 0]$, $-\tilde{Q} = \tilde{A}^T M + M \tilde{A} + \varsigma^2 \tilde{C}^T \tilde{C} + M \tilde{B} \tilde{B}^T M$, and

$$\tilde{Q} = \begin{bmatrix} \pi & \frac{\beta_1}{2} - \frac{\beta_1^2}{2} \\ \frac{\beta_1}{2} - \frac{\beta_1^2}{2} & \frac{\beta_1}{2} - 1 \end{bmatrix}$$

where $\pi = \frac{\beta_1^3}{2} - \frac{\beta_1^2}{4} + \beta_1 \beta_2 - \varsigma^2 + \beta_1^2 + 4\beta_2$. Define β_1, β_2 as in (23), and then $\tilde{Q}$ is a positive definite matrix. Thus one has

$$\dot{V}_i(s_i, z_i) \leq -|s_i|^{-1/2} \zeta_i^T \tilde{Q} \zeta_i < 0. \quad (27)$$

According to the similar definition in (9), (13) and Lemma 1, one has

$$\dot{V} \leq \sum \dot{V}_i(s_i, z_i) \leq -\tilde{\gamma} \sum V_i(s_i, z_i)^{1/2} \leq -\tilde{\gamma} V^{1/2} \quad (28)$$

where $\tilde{\gamma} = \frac{\lambda_{\min}\{\tilde{Q}\}}{\lambda_{\max}\{M\}}$. Following from the comparison

principle [23], when $t > \frac{\max_{1 \leq i \leq N} 2V_i(s_i(0), z_i(0))^{1/2}}{\tilde{\gamma}}$, we have $V_i(s_i, z_i) = 0$. The proof is thus completed. $\square$

Then, we further have the following results.

Theorem 4 Under the Assumption 1 and the topology contains a spanning tree, the super-twisting second-order sliding mode controller (20) guarantees that the MAS (1) can achieve consensus in finite-time.

4. Simulations

In this section, some examples are given to show the effectiveness of proposed results. A second-order nonlinear MAS with 10 agents is considered with its communication topology being in Fig. 1. Obviously, there is a directed spanning tree contained in this topology.

Fig. 1 Topology of MAS

The agent dynamics are described as follows:

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = f_i(t, x_i(t), v_i(t)) + u_i(t) \end{cases} \quad (29)$$

where $f(t, x_i(t), v_i(t)) = \sin(x_i(t)) + \sin(v_i(t))$, and $|f(t, x_i(t), v_i(t))| \leq 2$, and the Assumption 1 is obviously satisfied. Then we choose $\varsigma = 1$. The initial states of the MAS are chosen randomly.

(i) Consensus with second-order sliding mode observer and first-order sliding mode controller

According to Lemma 3, choose $\alpha_1 = 3, \alpha_2 = 8$, and the designed second-order sliding mode observer is given as follows:

$$\begin{cases} \dot{\hat{x}}_i = \hat{v}_i + 3|x_i - \hat{x}_i|^{1/2} \text{sgn}(x_i - \hat{x}_i) \\ \dot{\hat{v}}_i = f_i(t, \hat{x}_i, \hat{v}_i) + u_i + 8\text{sgn}(x_i - \hat{x}_i) \end{cases} \quad (30)$$

with the designed first-order sliding mode controller as in (15). Let $\beta = 1$, then

$$u_i = \sum_{j \in N_i} a_{ij}(\hat{v}_j - \hat{v}_i) - (1 + 2)\text{sgn}(s_i) \quad (31)$$

where $s_i = \hat{v}_i - \sum_{j \in N_i} a_{ij}(\hat{x}_j - \hat{x}_i)$.

The simulation results are shown in Fig. 2 and Fig. 3. Fig. 2 represents the observation errors of position and velocity of agents. One can find that the second-order sliding mode observer (30) can observe the velocity very well in finite time. For the other two control cases, we omit the similar simulation results. In Fig. 3(a) and Fig. 3(b), one can see under the controller (31), the positions and velocities of the MASs can achieve consensus. In Fig. 3(c), the sliding mode surfaces converge to the same point which lead to the consensus. It is shown in Fig. 3(d) that there exists a serious chattering problem in the control input.

(a) Position observation errors

(b) Velocity observation errors

Fig. 2 Observer errors of MASs

(a) Position

(b) Velocity

Fig. 3 Consensus with the first-order SMC in (31)

(ii) Consensus with sliding mode controller based on boundary layer

We consider the sliding mode controller based on the boundary layer as given in (18). Choose the boundary layer thickness as $\Delta = 0.01$ and $\beta = 1$, then

$$u_i = \sum_{j \in N_i} a_{ij}(\hat{v}_j - \hat{v}_i) - (1 + 2)\text{sat}(s_i). \quad (32)$$

The consensus results are shown in Fig. 4(a) and Fig. 4(b). One can find that the MAS (29) achieve consensus under the controller (32). Moreover, the sliding mode functions also achieve consensus as shown in Fig. 4(c). Fig. 4(d) shows that the sliding mode controller based on the boundary layer can effectively reduce chattering by comparison with the control input of first-order SMC in Fig. 3(d).

Fig. 4 Consensus with the boundary layer SMC in (32)

(iii) Consensus with second-order sliding mode controller

In order to balance the accuracy and chattering, we consider the super-twisting second-order sliding mode controller (20), choose the parameters as $\beta_1 = 3, \beta_2 = 9$ according to Theorem 3, then the controller can be written as follows:

$$\begin{cases} u_i = \sum_{j \in N_i} a_{ij}(\hat{v}_j - \hat{v}_i) - 3|s_i|^{1/2} \text{sgn}(s_i) + z_i \\ \dot{z}_i = -4\text{sgn}(s_i) \end{cases} \quad (33)$$

Obviously, the positions and velocities can achieve the consensus under the controller (30) as shown in Fig. 5(a) and Fig. 5(b).

In Fig. 5(c), the sliding mode functions converge to the origin. The control inputs are shown in Fig. 5(d), and one can see that the chatting is greatly reduced by the second-order SMC. Compared with the previous two controllers, it has much more better control performances both in overcoming the chatting problem and accuracy.

Fig. 5 Consensus with the second-order SMC in (33)

In order to further explain the advantages of the proposed second-order SMC in this paper, we compare it with the results in [8] where the second-order consensus tracking problem of uncertain multi-vehicle systems without velocity measurements is considered. The results are shown in Fig. 6(a) and Fig. 6(b). Compared with the consensus results as shown in Fig. 5(a) and Fig. 5(b), it is obviously that the convergent speed and robustness under our proposed second-order SMC are better than that of the consensus algorithm in [8].

Fig. 6 Consensus with the nonlinear robust controller in [8]

5. Conclusions

The second-order consensus problem of a class of nonlinear MASs is investigated via sliding mode observer and controller in this paper. Due to the advantages of sliding mode control, discontinuous sliding mode protocols are designed to achieve consensus in spite of the nonlinear dynamics. It is shown that under the designed interacted sliding mode surface and fully distributed protocol based on the local state information of neighboring agents the current agent can be drove to the desired steady state. Finally,

some simulations illustrate the effectiveness of the proposed schemes.

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