Journal of Systems Engineering and Electronics
Vol. 28, No. 4, August 2017, pp.793 – 800

Optimization of post-warranty sequential inspection for second-hand products

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Abstract: This paper considers an optimal sequential inspection schedule for a second-hand product after that the free nonrenewable warranty is expired. The length of warranty is prespecified and during the warranty period, the product is minimally repaired by the dealer when it fails. Following the expiration of the non-renewing warranty, the product is inspected and upgraded sequentially a fixed number of times at the expenses of the customer. At each inspection, the failure rate of the product is reduced proportionally so that the product is upgraded. The product is assumed to deteriorate as it ages and the replacement of the product occurs when a fixed number of inspections are rendered. In addition, the intervals between two successive inspections are assumed to decrease monotonically. The main objective of this paper is to determine the optimal improvement level to upgrade the product at each inspection so that the expected maintenance cost during the life cycle of the product is minimized from the perspective of the customer. Under the given cost structures, we derive an explicit formula to obtain the expected maintenance cost incurred during the life cycle of the product and discuss the method to find the optimal level of the improvement analytically in case the failure times follow the Weibull distribution. Numerical results are analyzed to observe the impact of relevant parameters on the optimal solution.

Keywords: second-hand product, sequential inspection, upgrade action, life cycle of the product, expected maintenance cost, free non-renewable warranty.

DOI: 10.21629/JSEE.2017.04.19

1. Introduction

As the product becomes more complex and multifunctional, the purchasing price becomes quite expensive and consequently the second-hand product would be an option for many customers. It is also reasonable to assume that many products keep deteriorating as it ages and so the second-hand product is likely to have more frequent failures during its life cycle compared to the new product. Due to a growing demand for the second-hand product, the best possible maintenance strategy in order to keep the product operating without failures as long as possible becomes a very crucial decision for both the dealer and the customer to make. At today's market, it becomes a common practice that the dealer offers a certain kind of warranty even for the second-hand product at its sale. In this respect, many researchers are interested in finding the best possible maintenance policy to optimize the product performance and to minimize the maintenance cost under several types of warranty policies.

Although there have been extensive researches regarding the optimal maintenance policies under several different types of warranties in case of the new product, the maintenance strategy regarding the second-hand product has very limited attention in the literature so far. As for many different types of warranty policies, Blischke and Murthy provided excellent references [1,2]. Reference [3] was considered the first paper to deal with the warranty of the second-hand products and later Chattopadhyay et al. also studied many aspects of free and pro-rata repair/replacement warranties for second-hand products and discussed the improvement of the product performance under certain cost structures [4,5]. Pongpech et al. considered an upgrade action for the used equipment with a fixed length of lease [6] and Shafiee et al. derived an optimal upgrade level by developing a stochastic model under certain profit structure [7]. Shafiee et al. also obtained an optimal improvement level under two reliability improvement models so that the dealer can be profitable [8] and proposed the optimal upgrade strategies under which the upgrading cost at the sale of the second-hand product can be compensated with the reduction of the expected war-

Manuscript received January 13, 2017.
*Corresponding author.
This work was supported by the Research Base Construction Fund Support Program funded by Chonbuk National University in 2013, and the Mid-career Research Program (2016R1A2B4010080) through NRF Grant funded by MEST.

ranty cost from the dealer's perspective [9]. More recently, Kim et al. obtained an optimal maintenance level for the second-hand product [10] when a periodic inspection was provided by the dealer while the warranty was in effective and an upgrade action was conducted at each inspection. More references regarding the optimal maintenance strategy and upgrade action for the second-hand product are given therein. Moreover, there are some discussions on two-dimensional warranty policies in the literature, which are based on two factors of age and usage upon the product failures. For instance, [11], [12], and [13], considered certain types of maintenance policies incorporating such two-dimensional warranty policies. The main objective of this paper is to find the optimal improvement level, by which the product is upgraded at each inspection for the second-hand product after the free non-renewable warranty is expired. Assuming certain cost structures, the expected total maintenance cost during the life cycle of the product is evaluated from the perspective of the customer and the optimality is obtained by minimizing the expected total maintenance cost. Although there exist many optimal maintenance policies based on periodic preventive maintenance during or following the warranty period for the new product, the sequential preventive inspection strategy for the second-hand product has not been studied often in the literature so far. The product is assumed to be deteriorating as it ages and thus, the product needs to be preventively inspected more often at a later stage of the life cycle of the product to reduce the chance of product failures. Therefore, it is more reasonable to consider an inspection schedule with variable intervals rather than with fixed intervals. In this respect, we study an optimal maintenance policy with a varying inspection interval in this paper, where the interval between two successive inspections becomes shorter as the product ages. No such maintenance policies dealing with the sequential inspection schedule for the second-hand product has been studied in the literature, to the best of our knowledge.

Consider a situation where a used car is sold with a free nonrenewable warranty. In such situation, the dealer could offer a free maintenance action such as sequential inspection during the warranty period to reduce the product failures. It is a common practice that the dealer offers only the minimal repair upon the product failure, not the replacement of the failed product. To formulate the expected total maintenance cost under our sequential inspection model with the free non-renewable warranty, we first define an adjusted failure rate at each inspection as a function of the improvement factor α and evaluate the expected number of failures during the maintenance period. The maintenance period is defined as the interval from the expiration of the warranty to the replacement of a new product. Consequently, the expected maintenance cost is used as an objective function for our optimization problem. It is true in general that the product failures occur more often as the product ages and so it is reasonable to inspect the product more frequently to reduce the product failures. More frequent inspections for the product are surely to increase the higher maintenance cost, but it could trade off with the lower chance of the product failures. This paper derives an optimal sequential inspection strategy to minimize the expected maintenance cost by determining the optimal improvement level, by which the product's failure rate is upgraded whenever the inspection is conducted, during the maintenance period following the warranty period.

The remainder of this paper is organized as follows. The nomenclature and model assumption are stated at the end of Section 1. In Section 2, the sequential preventive inspection model in conjunction with a free non-renewable warranty is described. In Section 3, we derive the adjusted failure rate at each inspection and compute the expected number of the product failures during the life cycle of the product. The expected total maintenance cost is evaluated under certain cost structures from the perspective of the customer. Section 4 considers an optimization problem to find an optimal inspection strategy under the given condition and discusses the existence of an optimal solution. Numerical examples are presented in Section 5 and finally, concluding remarks are given in Section 6.

Nomenclature

T is the time to failure of a product, $F_0(t)$ and $f_0(t)$ are life distribution and probability density function of T with no inspection /upgrade length of free non-renewable, ω is the warranty period, $h_0(t)$ is the failure rate function without periodic inspection/upgrade, where $h_0(t) = \dfrac{f_0}{1-F_0(t)}$, $h_\alpha(t)$ is the failure rate function after periodic inspection/upgrade action, x is the age of the second-hand product when purchased, τ is the length of the first inspection interval following the expiration of the warranty period, α is the improvement level, δ is the interval reduction factor for sequential inspection, $c_l(\alpha, x)$ is the unit inspection/upgrade cost with the improvement level of α. $\overline{c}$ is the unit inspection/upgrade cost, c_m is the unit minimal repair cost, c_f is the unit failure cost, $N_\omega(x)$ and $N_\omega(x,\alpha,n,\delta)$ are total numbers of failures during and following warranty period, respectively, $C_\omega(x,\alpha,n,\delta)$ is the total maintenance cost during the product's life cycle, $\mathrm{E}[R_\omega(x,\alpha,n,\delta)]$ is the expected maintenance cost rate.

Assumptions

(i) The age of a used product equals x.

(ii) The length of the warranty period is fixed at ω.

(iii) The inspection is rendered sequentially $(n-1)$

times at $X+\omega+\tau, x+\omega+(1+\delta)\tau, \ldots, x+\omega+(1+\delta+\cdots+\delta^{n-2})\tau$.

(iv) At each inspection, the product is upgraded by reducing the failure rate proportionally to the improvement level α.

(v) Only the minimal repair is rendered at each failure either during the warranty period or following the warranty period.

(vi) Replacement by a new one occurs at $t = x+\omega+(1+\delta+\cdots+\delta^{n-1})\tau$, at the nth inspection time.

2. Sequential inspection model under free non-renewable warranty

The sequential inspection model that we consider in this paper works as follows. At the sale of the second-hand product of the age x, the dealer offers a free non-renewable warranty of fixed length of ω. Thus, a minimal repair is rendered by the dealer on each product failure during the warranty period. Once the warranty is expired, the customer is responsible for all the costs incurring during the maintenance period, which follows the warranty period. To reduce the chance of the product failures during the maintenance period, a sequential preventive inspection is scheduled so that the length of intervals between two successive inspections becomes shorter as the number of inspections increases. Let τ be a positive number and δ be a number between 0 and 1, exclusively, and we assume that the first inspection is given τ time units after the warranty expires. Thus, the sequential inspections are rendered at times $x+\omega+\tau, x+\omega+(1+\delta)\tau, \ldots, x+\omega+(1+\delta+\cdots+\delta^{n-2})\tau$.

At the nth inspection at $t = x+\omega+(1+\delta+\cdots+\delta^{n-1})\tau$, the product is replaced by a new one. Here, τ denotes the waiting time until the first inspection occurs following the warranty and δ is a fixed constant by which each inspection interval is shortened. Note that the inspection intervals between two successive inspections are getting shorter by the multiple factor of δ. At each inspection, the product is upgraded by reducing its failure rate, which is dependent on the improvement level α. The product is replaced by a new one at the nth inspection and the life cycle of the second-hand product ends.

Fig. 1 illustrates the adjusted failure rate under our sequential inspection model, which is reproduced from [10] for convenience.

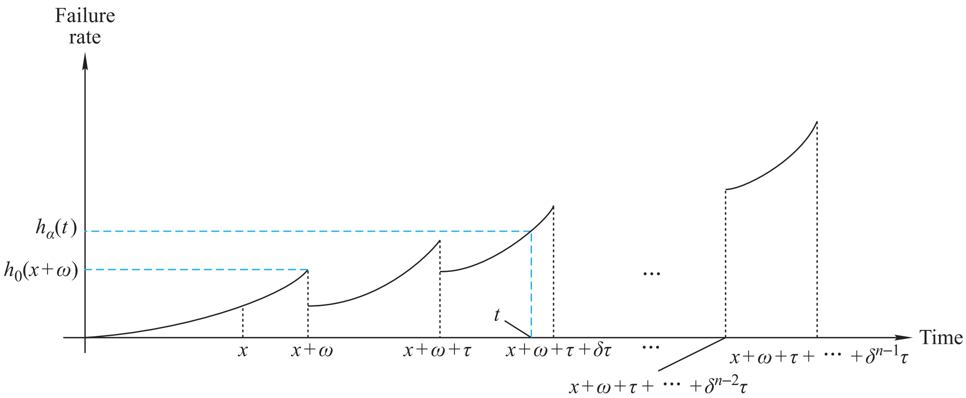

Fig. 1 Diagram for failure rate reduction at sequential inspection/upgrade

As seen in Fig. 1, the failure rate of the product is reduced proportionally to the improvement level α at each inspection time $x+\omega+\tau, x+\omega+(1+\delta)\tau, \ldots, x+\omega+(1+\delta+\cdots+\delta^{n-2})\tau$.

Fig. 2 describes the warranty period and maintenance period with the sequential inspection scheme.

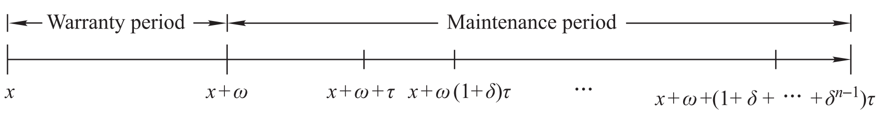

Fig. 2 Sequential inspection scheme for second-hand product

3. Product upgrade and evaluation of maintenance cost

In this section, we first obtain the mathematical formula to express the failure rate of the product during the maintenance period, which is adjusted at each sequential inspection. By reducing the failure rate proportionally to the improvement level α at each inspection, the failure rate of the product is adjusted so that the number of failures can be decreased. Then, by the consideration of the cost model we formulate in this paper we derive the expected total warranty cost from the customer's perspective which incurs during the maintenance period following the expiration of the free non-repairable warranty. All the product failures during and following the warranty period are only minimally repaired so that the failure rate remains the same as that just prior to its failures.

3.1 Upgrade action at sequential inspection

Let x denote the age of the second-hand product at the sale and $h_0(t)$ be the initial failure rate of the product with no previous upgrade action. It implies that the failure rate of the second hand product is equal to $h_0(x)$ when it is purchased by the customer. Since only minimal repairs are conducted on each product failure during the warranty period, no upgrade action is taken until the warranty is expired. It follows that the failure rate of the product is equal to $h_0(t)$ for $x+\omega \leqslant t < x+\omega+\tau$ where ω is the length of the warranty period and τ is a fixed positive number. Let $h_\alpha(t)$ denote the adjusted failure rate depending on the improvement level α. Then, $h_\alpha(t) = h_0(t)$ on the interval $[x+\omega, x+\omega+\tau)$ and $h_\alpha((x+\omega+\tau)^-) = h_0(x+\omega+\tau)$. Here $h_\alpha(t^-) = \lim\limits_{z\to 0} h_\alpha(t-\epsilon)$.

The first inspection is rendered at $t = x+\omega+\tau$ and its failure rate, denoted by $h_\alpha(x+\omega+\tau)$, is adjusted to

$$h_\alpha(x+\omega+\tau) = \alpha h_0(x+\omega+\tau) + (1-\alpha)h_0(x+\omega). \quad (1)$$

Equation (1) states that as the value of α gets closer to 1, the upgraded failure rate becomes closer to the original failure rate and so the product receives very small upgrade in terms of its failure rate. On the other hand, as α gets closer to 0, the failure rate is reduced to that the existing τ time units previous to the inspection. That is, when $\alpha = 1$, there is no upgrade of the product and when $\alpha = 0$, the failure rate is reduced to that of $t = x+\tau$, which is considered as the best possible upgrade. Thus, the smaller the value of α is, the more the upgrade of the product is achieved. We assume that although the failure rate is reduced by the upgrade action at each inspection, the shape of failure rate remains the same as the one in the previous inspection interval.

For $x+\omega+\tau \leqslant t < x+\omega+(1+\delta)\tau$, the upgraded failure rate can be derived as

$$H_\alpha(t) = \alpha h_\alpha((x+\omega+\tau)^-) + (1-\alpha)h_\alpha(x+\omega)+$$
$$[h_0(t-\tau) - h_0(x+\omega)] = \alpha[h_0(x+\omega+\tau) - h_0(x+\omega)]+$$
$$h_0(x+\omega) + h_0(t-\tau) - h_0(x+\omega) =$$
$$\alpha[h_0(x+\omega+\tau) - h_0(x+\omega)] + h_0(t-\tau).$$

At $t = (x+\omega+(1+\delta)\tau)^-$, the failure rate is upgraded as

$$H_\alpha((x+\omega+(1+\delta)\tau)^-) = \alpha[h_0(x+\omega+\tau)-$$
$$h_0(x+\omega)] + h_0(x+\omega+\delta\tau).$$

Applying the similar argument, the upgraded failure rate on the interval $[x+\omega+(1+\delta)\tau, x+\omega+(1+\delta+\delta^2)\tau]$ becomes

$$h_\alpha(t) = \alpha h_\alpha((x+\omega+(1+\delta)\tau)^-) + (1-\alpha)\cdot$$
$$(1+\delta)\tau) - h_\alpha(x+\omega+\tau) + (h_0(t-h_0(x+\omega)) =$$
$$\alpha^2((h_0(x+\omega+\delta\tau) - h_0(x+\omega))+$$
$$\alpha h_0(x+\omega+\delta\tau) + \alpha(1-\alpha)\cdot$$
$$((h_0(x+\omega+\tau) - h_0(x+\omega)) + (1-\alpha)h_0(x+\omega) =$$
$$\alpha(h_0(x+\omega+\tau) + h_0(x+\omega+\delta\tau)-$$
$$2h_0(x+\omega)) + h_0(t-(1+\delta)\tau).$$

Repeating the similar arguments in succession, the failure rate upgraded sequentially at each inspection during the maintenance period following the expiration of the warranty period can be expressed as follows.

For $k = 1, 2, \ldots, n-1$, $h_\alpha(t)$ can be expressed in general as

$$h_\alpha(t) = \begin{cases} h_0(t), \quad x+\omega \leqslant t < x+\omega+\tau \\ \alpha\left\{\sum\limits_{i=1}^{k} h_0(x+\omega+\delta^{i-1}\tau) - kh_0(x+\omega)\right\} + \\ \qquad h_0\left(t - \dfrac{1-\delta^k}{1-\delta}\tau\right) \\ x+\omega+\dfrac{1-\delta^k}{1-\delta}\tau \leqslant t < x+\omega+\dfrac{1-\delta^{k+1}}{1-\delta}\tau \end{cases} . \quad (2)$$

Note that the product is replaced at the nth inspection time and $1+\delta+\delta^2+\cdots+\delta^{k-1} = \dfrac{1-\delta^k}{1-\delta}$ for $0 < \delta < 1$. Equation (2) gives the general formula to evaluate the upgraded failure rate of the product on each inspection interval under the proposed sequential inspection model when the improvement level equals α.

3.2 Cost model and expected maintenance cost

Let $c_1(\alpha, x)$ denote the inspection cost incurring at each inspection when the improvement level equals α and the age of the product of the second-hand product at its purchase equals x. By assuming that the inspection cost is an increasing function of x and is a decreasing function of α, which is reasonable in practice, we consider the following form of the cost model as given below.

$$c_1(\alpha, x) = \overline{c}(1-\alpha)^\theta e^{\gamma x} \quad (3)$$

where $\overline{c}$ is the fixed base cost charged for each inspection when an upgrade action is taken and γ and θ are the arbitrary positive numbers. Equation (3) implies that if no upgrade is taken, that is $\alpha = 1$, the inspection cost becomes 0 and no cost incurs. As α decreases, the product is upgraded more and the inspection cost increases. Similar forms of cost models have been discussed.

Let c_m and c_f be the unit minimal repair cost and the unit failure cost, respectively. Under the free non-repairable warranty, the customer is responsible for the failure cost during the warranty period and both minimal repair cost and failure cost during the maintenance period. Thus, to evaluate the total maintenance cost from the customer's perspective, we need to find the number of failures, which can be evaluated by utilizing the failure rates, given in (2). Let $N_\omega(x)$ and $N_\omega(\alpha, x, n, \delta)$ be the expected number of failures occurring during the warranty period and the maintenance period until the nth inspection, respectively. Then, we have

$$N_\omega(x) = \int_x^{x+\omega} h_0(t)\mathrm{d}t$$

$$N_\omega(\alpha, x, n, \delta) = \int_{x+\omega}^{x+\omega+\frac{1-\delta^n}{1-\delta}\tau} h_\alpha(t)\mathrm{d}t = \sum_{k=0}^{n-1} \int_{x+\omega+\frac{1-\delta^k}{1-\delta}\tau}^{x+\omega+\frac{1-\delta^{k+1}}{1-\delta}\tau} h_\alpha(t)\mathrm{d}t \tag{4}$$

where $h_\alpha(t)$ is the upgraded failure rate which is defined in (2). Thus, the expected total maintenance cost during the life cycle of the product, denoted by $\mathrm{E}[C_\omega(x, \alpha, n, \delta)]$, can be expressed as

$$\mathrm{E}[C_\omega(x, \alpha, n, \delta)] = c_f N_\omega(x) + (n-1)c_1(\alpha, x) + (c_m + c_f)N_\omega(\alpha, x, n, \delta) \tag{5}$$

where $N_\omega(x)$ and $N_\omega(\alpha, x, n)$ are defined as in (4). Note that the product is replaced by a new one without the upgrade action at the nth inspection and thus, the inspection cost occurs $(n-1)$ times. Note also that the failure cost is at the expense of the customer even when the free non-renewable warranty is in effective.

In the next section, we determine the optimal value of α which minimizes $\mathrm{E}[C_\omega(x, \alpha, n, \delta)]$ under certain situations.

4. Optimal sequential inspection policy

The optimization problem under our proposed sequential inspection model can be stated as follows.

Find α^* satisfying

$$\mathrm{E}[C_\omega(x, \alpha^*, n, \delta)] = \min_\alpha \mathrm{E}[C_\omega(x, \alpha, n, \delta)]. \tag{6}$$

In general, as the value of α decreases with other parameters fixed, the upgrade level of the product increases and thus, the expected number of failures during the life cycle is also expected to decrease. However, as seen in (3) the inspection cost increases as α decreases, which implies that the expected maintenance cost may not be a monotonic function of α. Thus, we need to find the value of α at which $\mathrm{E}[C_\omega(x, \alpha, n, \delta)]$ is minimized. By taking the first derivative of $\mathrm{E}[C_\omega(x, \alpha, n, \delta)]$, given in (5), with respect to α and setting it equal to 0, we have

$$(n-1)\frac{\mathrm{d}}{\mathrm{d}\alpha}c_1(\alpha, x) + (c_m + c_f)\frac{\mathrm{d}}{\mathrm{d}\alpha}N_\omega(\alpha, x, n, \delta) = (n-1)[-\bar{c}\theta(1-\alpha)^{\theta-1}\mathrm{e}^{\gamma x}] + c_f)\cdot \sum_{k=0}^{n-1} \int_{x+\omega+\frac{1-\delta^k}{1-\delta}\tau}^{x+\omega+\frac{1-\delta^{k+1}}{1-\delta}\tau} \frac{\mathrm{d}}{\mathrm{d}\alpha}h_\alpha(t)\mathrm{d}t. \tag{7}$$

Note that $\frac{\mathrm{d}}{\mathrm{d}\alpha}h_\alpha(t) = \sum_{i=1}^{k} h_0(x+\omega+\delta^{i-1}\tau) - kh_0(x+\omega)$, $k = 1, 2, \cdots, n-1$ for $x + \omega + \frac{1-\delta^k}{1-\delta}\tau \leqslant t < x + \omega + \frac{1-\delta^{k+1}}{1-\delta}\tau$ and it does not depend on t. Thus, (7) can have the simpler expression as follows:

$$(n-1)[-\bar{c}\theta(1-\alpha)^{\theta-1}\mathrm{e}^{\gamma x}] + (c_m + c_f)\cdot \sum_{k=1}^{n-1}[\delta^k\tau]\left\{\sum_{i=1}^{k} h_0(x+\omega+\delta^{i-1}\tau) - kh_0(x+\omega)\right\} = 0. \tag{8}$$

From (8), we obtain the second derivative of $\mathrm{E}[C_\omega(x, \alpha, n, \delta)]$ with respect to α as

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\mathrm{E}[C_\omega(x, \alpha, n, \delta)] = (n-1)\bar{c}\theta(\theta-1)(1-\alpha)^{\theta-2}\mathrm{e}^{\gamma x}$$

which is positive for $\theta > 1$ and $0 < \alpha < 1$. It follows that $\mathrm{E}[C_\omega(x, \alpha, n, \delta)]$ is a convex function of α for $\theta > 1$ and thus the unique solution for α exists in (8).

The optimal inspection level, denoted by α^*, can be obtained by solving (8) for α as follows:

$$\alpha^* = 1 - \left\{\frac{(c_m + c_f)\sum_{k=1}^{n-1}[\delta^k\tau]\left\{\sum_{i=1}^{k} h_0(x+\omega+\delta^{i-1}\tau) - kh_0(x+\omega)\right\}}{(n-1)\bar{c}\theta\mathrm{e}^{\gamma x}}\right\}^{\frac{1}{\theta-1}} \tag{9}$$

for $\theta > 1$ and $0 < \alpha < 1$. Note also that the inspection cost is affected by not only θ and γ, but also by the improvement level and the initial age of the product at purchase as shown in (3). The values of τ and δ parameterize the length of maintenance period following the expiration of the free non-renewing warranty.

5. Optimal solution for Weibull failure distribution

In this section, we consider a special case when the failure distribution of the product follows a Weibull distribution for the purpose of finding the optimal solution for α with other parameters fixed.

5.1 Weibull distribution case

To exemplify our proposed method, we consider a Weibull distribution as the failure distribution of the second-hand product, of which the distribution and failure rate function are given as follows:

$$F_0(t) = 1 - \exp(-(\eta t)^\beta), \quad \eta, \beta > 0$$

$$h_0(t) = \eta\beta(\eta t)^{\beta-1} \tag{10}$$

where η and β are the scale and shape parameters of the Weibull distribution. We first consider the sequential inspection model with $n = 3$ under which the product is inspected and upgraded twice and at the third inspection the product is replaced by a new one without an upgrade action. In this case, the upgraded failure rate can be expressed as

$$h_\alpha(t) = \begin{cases} h_0(t), \quad x+\omega \leqslant t < x+\omega+\tau \\ \alpha A + h_0(t-\tau), \quad x+\omega+\tau \leqslant \\ \qquad t < x+\omega+(1+\delta)\tau \\ \alpha B + h_0(t-(1+\delta)\tau), \quad x+\omega+ \\ \qquad (1+\delta)\tau \leqslant t < x+\omega+(1+\delta+\delta^2)\tau \end{cases}$$

where $A = h_0(x+\omega+\tau) - h_0(x+\omega)$ and $B = h_0(x+\omega+\tau) + h_0(x+\omega+\delta\tau) - 2h_0(x+\omega)$. By utilizing the formula given in (9), the solution for α can be obtained as follows:

$$\alpha^* = 1 - \left\{ \frac{(c_m+c_f)\displaystyle\sum_{k=1}^{2}[\delta^k\tau\eta\beta]\left\{\sum_{i=1}^{k}[\eta(x+\omega+\delta^{i-1}\tau)]^{\beta-1} - k[\eta(x+\omega)]^{\beta-1}\right\}}{(n-1)\overline{c}\theta \mathrm{e}^{\gamma x}} \right\}^{\frac{1}{\theta-1}}. \tag{11}$$

For simplicity of calculations, we consider a special case when $\beta = 2$ for the Weibull distribution. Then, for $n = 3$, we obtain

$$\alpha^* = 1 - \left\{ \frac{(c_m+c_f)\eta^2\delta\tau^2(1+\delta(1+\delta))}{\overline{c}\theta \mathrm{e}^{\gamma x}} \right\}^{\frac{1}{\theta-1}}. \tag{12}$$

Applying the similar arguments for $n = 4$, the optimal solution for α can be obtained as

$$\alpha^* = 1 - \left\{ \frac{(c_m+c_f)\eta^2\delta\tau^2(1+\delta(1+\delta)+\delta^2(1+\delta+\delta^2))}{\overline{c}\theta \mathrm{e}^{\gamma x}} \right\}^{\frac{1}{\theta-1}}. \tag{13}$$

By repeating the same arguments successively, the unique optimal solution for α can be obtained for arbitrary n for the special case when the failure of the product follows a Weibull distribution with the shape parameter of $\beta = 2$. In this case, the solution for α is given as

$$\alpha^* = 1 - \left\{ \frac{(c_m+c_f)\eta^2\delta\tau^2\left(\displaystyle\sum_{i=2}^{n}\delta^{i-2}\frac{(1-\delta^{i-1})}{1-\delta}\right)}{\overline{c}\theta \mathrm{e}^{\gamma x}} \right\}^{\frac{1}{\theta-1}} \tag{14}$$

for $n \geqslant 2$ in general. Although the optimal solution for α is readily obtainable for the Weibull distribution with $\beta = 2$, the explicit solution for $\beta \geqslant 3$ could be extremely difficult to obtain analytically due to the complex functional forms and so we may need some numerical methods to solve the optimization problem.

5.2 Numerical examples

To analyze the impact of other relevant parameters on the optimal solution for α, we present numerical results based on several combinations of parameters in this subsection. We fix $\omega = 3, \tau = 0.5, \gamma = 1$ and all the time units are measured in year. As for the unit costs, we take $\overline{c} = 100, c_m = 200$ and $c_f = 50$ and the failure distribution is assumed to follow a Weibull distribution with $\beta = 2$ and $\mu = 0.4$. We choose these values so that the failure time for the product is approximately equal to 2.216, which is smaller than the warranty period. We also choose $n = 4$ so that the product is replaced at the 4th inspection.

Table 1 presents the optimal values of the improvement level, denoted by α^* for various combinations of x, δ and θ, which are obtained by using (13). As x increases, α^* decreases steadily, but slowly for fixed values of θ and δ, which implies that the older the product is when it is purchased, the higher level of upgrade at each inspection is needed to optimize the maintenance cost. Table 1 also shows that the value of α^* decreases steadily as δ decreases when the values of θ and x are fixed. It indicates that when the sequential inspection is performed more frequently, the higher level of upgrade is more desirable to minimize the maintenance cost from the customer's perspective after the warranty is expired. Note that in case $\delta = 1$ the sequential inspection model is reduced to the periodic model.

Table 1 Optimal improvement level α^* for various combinations of x and δ for given θ

θ	x	δ			
		1	0.8	0.6	0.4
1.5	1	0.086 6	0.017 6	0.003 0	0.000 4
	1.5	0.031 9	0.006 5	0.001 1	0.000 2
	2	0.011 7	0.002 4	0.000 4	0.000 1
	2.5	0.004 3	0.000 9	0.000 1	0.000 0
	3	0.001 6	0.000 3	0.000 1	0.000 0
2.0	1	0.220 7	0.099 6	0.040 9	0.015 3
	1.5	0.133 9	0.060 4	0.024 8	0.009 3
	2	0.081 2	0.036 6	0.015 0	0.005 6
	2.5	0.049 3	0.022 2	0.009 1	0.003 4
	3	0.029 9	0.013 5	0.005 5	0.002 1
2.5	1	0.314 7	0.185 1	0.102 2	0.053 1
	1.5	0.225 5	0.132 7	0.073 3	0.038 0
	2	0.161 6	0.095 0	0.052 5	0.027 3
	2.5	0.115 8	0.068 1	0.037 6	0.019 5
	3	0.083 0	0.048 8	0.026 9	0.014 0
3.0	1	0.383 6	0.257 6	0.165 0	0.101 0
	1.5	0.298 8	0.200 7	0.128 5	0.078 6
	2	0.232 7	0.156 3	0.100 1	0.061 2
	2.5	0.181 2	0.121 7	0.078 0	0.047 7
	3	0.141 1	0.094 8	0.060 7	0.037 1

6. Concluding remarks

In order to reduce the product failures, it is a reasonable practice for the customer to inspect and to upgrade the product more frequently as it ages. Although there exist a number of optimal maintenance policies for the product under a variety of situations, most of them consider a periodic maintenance policy rather than a sequential scheme.

Since the product is assumed to have an increasing failure rate, the chance of product failures increases as the product ages. Thus, an optimal maintenance policy with a varying inspection schedule is more desirable in practice than the one with a periodic inspection schedule. The sequential inspection schedule that we consider in this paper could be more applicable in practice because of its flexibility. Although there exist abundant references for the optimal maintenance policies with periodic inspection schedules, very few researches have been done for sequential inspection schedules, especially for the second-hand product.

This study deals with an optimal sequential inspection/upgrade policy for the second-hand product after the free non-renewable warranty is expired. Under our proposed sequential model, the length of the inspection interval is decreased by a multiple of δ, which assumes the value is between 0 and 1. If we take $\delta = 1$ the model is reduced to the periodic maintenance model. During the free non-renewing warranty period, the product is only minimally repaired by the dealer when it fails. Following the expiration of the warranty, the product is inspected sequentially a fixed number of times at the expenses of the customer. At each inspection, the product is upgraded as a function of the improvement level by reducing its failure rate. The product is assumed to deteriorate as it ages and the replacement of the product occurs when a fixed number of inspections are rendered. In addition, the intervals between two successive inspections are assumed to decrease monotonically.

The main objective of this paper is to determine the optimal improvement level minimizing the maintenance cost when the sequential inspection/upgrade scheme is adopted after the warranty is expired. The proposed model subsumes the periodic maintenance policy and the inspection scheme of a new product as the special cases by taking $\delta = 1$ and $x = 0$, respectively. When the failure times of the product follow a Weibull distribution with the shape parameter of two, the unique solution for the decision variable is obtained explicitly in general. However, when the shape parameter is greater than two, the explicit solution may not be possible to obtain analytically in general and so we will have to use the suitable numerical methods. Numerical comparison gives several interesting observations regarding the dependency of the optimal improvement level on the length of the inspection interval and the age of second-hand products when purchased, respectively.

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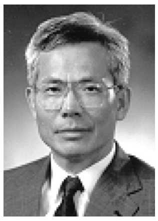

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