Journal of Systems Engineering and Electronics
Vol. 28, No. 6, December 2017, pp.1152 – 1161

Cartesian product over interval valued intuitionistic fuzzy sets

Jianming Xie* and Sanyang Liu

School of Mathematics and Statistics, Xidian University, Xi'an 710071, China

Abstract: The intuitionistic fuzzy set (IFS) based on fuzzy theory, which is of high efficiency to solve the fuzzy problem, has been introduced by Atanassov. Subsequently, he pushed the research one step further from the IFS to the interval valued intuitionistic fuzzy set (IVIFS). On the basis of fuzzy set (FS), the IFS is a generalization concept. And the IFS is generalized to the IVIFS. In this paper, the definition of the sixth Cartesian product over IVIFSs is first introduced and its some properties are explored. We prove some equalities based on the operation and the relation over IVIFSs. Finally, we present one geometric interpretation and a numerical example of the sixth Cartesian product over IVIFSs.

Keywords: intuitionistic fuzzy sets (IFS), Cartesian product, operation, geometric interpretation, interval valued intuitionistic fuzzy set (IVIFS).

DOI: 10.21629/JSEE.2017.06.13

1. Introduction

In 1986, the theory of intuitionistic fuzzy sets (IFS) was presented initially by Atanassov [1] and a series of operations and concepts is defined in [2]. Later, the interval valued intuitionistic fuzzy sets (IVIFS) [3] was presented by Atanassov in 1989, and it has achieved a tremendous amount of research and development in various fields. For example, Zeng and Hu studied the necessity operator and the possibility operator of IVIFSs [4]. Zeng and Wu proposed and proved some theorems over the first Cartesian product of IVIFSs [5]. Many scholars applied IFS and IVIFS to multiple attribute decision making and pattern recognition. Yager, Yuan and Li discussed the cut set characteristics of IVIFS in [6 – 8], and Xu and Zhang applied IVIFS to pattern recognition [9 – 11]. Wei, Xu and Li also applied IVIFSs to decision-making analysis [12 – 16]. Chen proposed some methods for multiple criteria decision analysis based on IVIFSs [17,18]. Lei and Zhang made some researches on interval-valued intuitionistic fuzzy reasoning [19,20]. Different similarity measures and their applications were studied in [21,22]. For distance measures, further research was done by Khalid and Abbas [23].

Atanassov defined five versions of Cartesian products of two IFSs. The sixth cartesian product over IFSs was defined by Velin Andonov in 2008 [24]. The seventh, eighth and ninth cartesian products over IFSs were defined by Annie Varghese and Sunny Kuriakose in 2012 [25], they also defined the tenth and eleventh Cartesian products over IFSs in the same year [26], that is to say $\times_7$, $\times_8$, $\times_9$, $\times_{10}$ and $\times_{11}$, respectively. And the corresponding equations for some operations and relations over IFSs were proved.

According to the comparison of interval-valued fuzzy sets (IVFS) and IFS, Atanassov introduced five kinds of Cartesian products of two IVIFSs after he had proposed IVIFS. As we all know, there are eleven kinds of Cartesian products of IFSs. However, there are only five types of Cartesian products of IVIFSs. Thus, it is urgent to put forward the Cartesian product of IVIFSs. This paper defines the sixth Cartesian product of the IVIFS. And some properties of the two IVIFSs are discussed.

The paper is structured as follows. Detailed definitions and various properties of IFS and IVIFS are briefly elaborated in Section 2. Some of the main results are shown in Section 3. We can see that the geometric interpretation of the sixth Cartesian product of IVIFS in Section 4. A numerical example is given in Section 5. Finally, we make a short conclusion in Section 6.

2. Preliminaries

In this section, several fundamental notions about IFS and IVIFS are introduced.

Definition 1 (IFS[2]) Let U be a universe. Then we can use the following formulas to define an IFS α:

$$\alpha = \{\langle \omega, \mu_\alpha(\omega), \nu_\alpha(\omega)\rangle \mid \omega \in U\}$$

where the function $\mu_\alpha(\omega)$ means

$$\mu_\alpha : U \to [0, 1]$$

Manuscript received October 21, 2016.
*Corresponding author.
This work was supported by the National Natural Science Foundation of China (61373174).

and the $\nu_\alpha(\omega)$ is denoted as

$$\nu_\alpha : U \to [0,1]$$

where $\mu_\alpha(\omega)$ evaluates the degree of membership for each $\omega \in U$, while $\nu_\alpha(\omega)$ implicates the degree of non-membership. For each $\omega \in U$, there are

$$0 \leqslant \mu_\alpha(\omega) + \nu_\alpha(\omega) \leqslant 1.$$

Below we will show the notion of an IVIFS, which is a generalization of the IFS and the IVFS.

Definition 2 (IVIFS[1]) An IVIFS α in U is defined as follows:

$$\alpha = \{\langle \omega, m_\alpha(\omega), n_\alpha(\omega)\rangle \mid \omega \in U\}$$

where

$$m_\alpha(\omega) \subset [0,1], n_\alpha(\omega) \subset [0,1]$$

are intervals, and the following inequality is hold for each $\omega \in U$:

$$\sup m_\alpha(\omega) + \sup n_\alpha(\omega) \leqslant 1.$$

Definition 3 [1] An IVIFS α in U is defined by the membership function

$$m_\alpha : U \to \mathrm{INT}[0,1],$$

the non-membership function

$$n_\alpha : U \to \mathrm{INT}[0,1]$$

where INT[0,1] denotes a set which includes all subsets of [0,1].

Proposition 1 [1] For all IVIFSs α and β the following equations hold:

$$\overline{\alpha} = \{\langle \omega, n_\alpha(\omega), m_\alpha(\omega)\rangle \mid \omega \in U\},$$

$$\alpha \cap \beta = \{\langle \omega, [\min(\inf m_\alpha(\omega), \inf m_\beta(\omega)), \min(\sup m_\alpha(\omega), \sup m_\beta(\omega))], [\max(\inf n_\alpha(\omega), \inf n_\beta(\omega)), \max(\sup n_\alpha(\omega), \sup n_\beta(\omega))]\rangle \mid \omega \in U\},$$

$$\alpha \cup \beta = \{\langle \omega, [\max(\inf m_\alpha(\omega), \inf m_\beta(\omega)), \max(\sup m_\alpha(\omega), \sup m_\beta(\omega))], [\min(\inf n_\alpha(\omega), \inf n_\beta(\omega)), \min(\sup n_\alpha(\omega), \sup n_\beta(\omega))]\rangle \mid \omega \in U\},$$

$$\alpha + \beta = \{\langle \omega, [\inf m_\alpha(\omega) + \inf m_\beta(\omega) - \inf m_\alpha(\omega) \cdot \inf m_\beta(\omega), \sup m_\alpha(\omega) + \sup m_\beta(\omega) - \sup m_\alpha(\omega) \cdot \sup m_\beta(\omega)], [\inf n_\alpha(\omega) \cdot \inf n_\beta(\omega), \sup n_\alpha(\omega) \cdot \sup n_\beta(\omega)]\rangle \mid \omega \in U\},$$

$$\alpha \cdot \beta = \{\langle \omega, [\inf m_\alpha(\omega) \cdot \inf m_\beta(\omega), \sup m_\alpha(\omega) \cdot \sup m_\beta(\omega)], [\inf n_\alpha(\omega) + \inf n_\beta(\omega) - \inf n_\alpha(\omega) \cdot \inf n_\beta(\omega), \sup n_\alpha(\omega) + \sup n_\beta(\omega) - \sup n_\alpha(\omega) \cdot \sup n_\beta(\omega)]\rangle \mid \omega \in U\},$$

$$\alpha @ \beta = \{\langle \omega, [\frac{\inf m_\alpha(\omega) + \inf m_\beta(\omega)}{2}, \frac{\sup m_\alpha(\omega) + \sup m_\beta(\omega)}{2}], [\frac{\inf n_\alpha(\omega) + \inf n_\beta(\omega)}{2}, \frac{\sup n_\alpha(\omega) + \sup n_\beta(\omega)}{2}]\rangle \mid \omega \in U\}.$$

Proposition 2 [1] For all IVIFSs α and β, there hold:

$$\alpha \subset \beta \Longleftrightarrow (\forall \omega \in U)(\sup m_\alpha(\omega) \leqslant \sup m_\beta(\omega)) \& (\inf m_\alpha(\omega) \leqslant \inf m_\beta(\omega)) \& (\sup n_\alpha(\omega) \geqslant \sup n_\beta(\omega)) \& (\inf n_\alpha(\omega) \geqslant \inf n_\beta(\omega)).$$

$$\alpha = \beta \Longleftrightarrow (\forall \omega \in U)(\sup m_\alpha(\omega) = \sup m_\beta(\omega)) \& (\inf m_\alpha(\omega) = \inf m_\beta(\omega)) \& (\sup n_\alpha(\omega) = \sup n_\beta(\omega)) \& (\inf n_\alpha(\omega) = \inf n_\beta(\omega)).$$

Definition 4 [4] The modal type operators for IVIFSs are given as follows:
the necessity operator

$$\Box\alpha = \{\langle \omega, m_\alpha(\omega), [\inf n_\alpha(\omega), 1 - \sup m_\alpha(\omega)]\rangle \mid \omega \in U\},$$

the possibility operator

$$\Diamond\alpha = \{\langle \omega, [\inf m_\alpha(\omega), 1 - \sup n_\alpha(\omega)], n_\alpha(\omega)\rangle \mid \omega \in U\}.$$

Definition 5 [1] Let α be an IVIFS, there are four equations which map an IVIFS to an IFS:

$$*_1\alpha = \{\langle \omega, \inf m_\alpha(\omega), \inf n_\alpha(\omega)\rangle \mid \omega \in U\},$$

$$*_2\alpha = \{\langle \omega, \inf m_\alpha(\omega), \sup n_\alpha(\omega)\rangle \mid \omega \in U\},$$

$$*_3\alpha = \{\langle \omega, \sup m_\alpha(\omega), \inf n_\alpha(\omega)\rangle \mid \omega \in U\},$$

$$*_4\alpha = \{\langle \omega, \sup m_\alpha(\omega), \sup n_\alpha(\omega)\rangle \mid \omega \in U\}.$$

3. Main results

Let U_1 and U_2 be two universes. α is an IVIFS in U_1, β is an IVIFS in U_2, i.e.

$$\alpha = \{\langle \omega, m_\alpha(\omega), n_\alpha(\omega)\rangle \mid \omega \in U_1\},$$

$$\beta = \{\langle \psi, m_\beta(\psi), n_\beta(\psi)\rangle \mid \psi \in U_2\}.$$

Definition 6 Cartesian product

$$\alpha \times_6 \beta = \{\langle\langle\omega,\psi\rangle, [\frac{\inf m_\alpha(\omega)+\inf m_\beta(\psi)}{2}, \frac{\sup m_\alpha(\omega)+\sup m_\beta(\psi)}{2}], [\frac{\inf n_\alpha(\omega)+\inf n_\beta(\psi)}{2}, \frac{\sup n_\alpha(x)+\sup n_\beta(\psi)}{2}]\rangle \mid \omega\in U_1 \& \psi\in U_2\}.$$

Since

$$0 \leqslant \frac{\inf m_\alpha(\omega)+\inf m_\beta(\psi)+\inf n_\alpha(\omega)+\inf n_\beta(\psi)}{2} \leqslant 1,$$

$$0 \leqslant \frac{\sup m_\alpha(\omega)+\sup m_\beta(\psi)+\sup n_\alpha(\omega)+\sup n_\beta(\psi)}{2} \leqslant 1,$$

the result is IVIFS over the universes $U_1 \times U_2$.

Proposition 3 α, β denote an IVIFS in U_1 and U_2, respectively, then

$$\overline{\overline{\alpha} \times_6 \overline{\beta}} = \alpha \times_6 \beta. \tag{1}$$

Proof Depend on Proposition 1 and Definition 6,

$$\overline{\overline{\alpha} \times_6 \overline{\beta}} = \overline{\overline{\{\langle\omega, m_\alpha(\omega), n_\alpha(\omega)\rangle \mid \omega\in U_1\}} \times_6 \overline{\{\langle\psi, m_\beta(\psi), n_\beta(\psi)\rangle \mid \psi\in U_2\}}} =$$

$$\overline{\{\langle\omega, n_\alpha(\omega), m_\alpha(\omega)\rangle \mid \omega\in U_1\} \times_6 \{\langle\psi, n_\beta(\psi), m_\beta(\psi)\rangle \mid \psi\in U_2\}} =$$

$$\{\langle\langle\omega,\psi\rangle, [\frac{\inf m_\alpha(\omega)+\inf m_\beta(\psi)}{2}, \frac{\sup m_\alpha(\omega)+\sup m_\beta(\psi)}{2}], [\frac{\inf n_\alpha(\omega)+\inf n_\beta(\psi)}{2}, \frac{\sup n_\alpha(\omega)+\sup n_\beta(\psi)}{2}]\rangle \mid \omega\in U_1 \& \psi\in U_2\} =$$

$$\alpha \times_6 \beta. \qquad \square$$

Proposition 4 Given two universes U_1 and U_2. α and β are IVIFSs in U_1, γ is an IVIFS in U_2. Then these equalities hold

$$(\alpha\cup\beta)\times_6\gamma = (\alpha\times_6\gamma)\cup(\beta\times_6\gamma) \tag{2}$$

$$(\alpha\cap\beta)\times_6\gamma = (\alpha\times_6\gamma)\cap(\beta\times_6\gamma) \tag{3}$$

$$\gamma\times_6(\alpha\cup\beta) = (\gamma\times_6\alpha)\cup(\gamma\times_6\beta) \tag{4}$$

$$\gamma\times_6(\alpha\cap\beta) = (\gamma\times_6\alpha)\cap(\gamma\times_6\beta). \tag{5}$$

Proof We shall prove (2) and (3). For (4) and (5), the proofs are analogous.

On the left side of (2), we have

$$(\alpha\cup\beta)\times_6\gamma = \{\langle\omega, [\max(\inf m_\alpha(\omega), \inf m_\beta(\omega)), \max(\sup m_\alpha(\omega), \sup m_\beta(\omega))], [\min(\inf n_\alpha(\omega), \inf n_\beta(\omega)), \min(\sup n_\alpha(\omega), \sup n_\beta(\omega))]\rangle \mid \omega\in U\}$$

$$\times_6\gamma = \{\langle\langle\omega,\psi\rangle, [\frac{\inf\max(m_\alpha(\omega), m_\beta(\omega))+\inf m_\gamma(\psi)}{2}, \frac{\sup\max(m_\alpha(\omega), m_\beta(\omega))+\sup m_\gamma(\psi)}{2}], [\frac{\inf\min(n_\alpha(\omega), n_\beta(\omega))+\inf n_\gamma(\psi)}{2}, \frac{\sup\min(n_\alpha(\omega), n_\beta(\omega))+\sup n_\gamma(\psi)}{2}]\rangle \mid \omega\in U_1 \& \psi\in U_2\}.$$

On the right side of (2), we have

$$(\alpha\times_6\gamma)\cup(\beta\times_6\gamma) = \{\langle\langle\omega,\psi\rangle, [\frac{\inf m_\alpha(\omega)+\inf m_\gamma(\psi)}{2}, \frac{\sup m_\alpha(\omega)+\sup m_\gamma(\psi)}{2}], [\frac{\inf n_\alpha(\omega)+\inf n_\gamma(\psi)}{2}, \frac{\sup n_\alpha(\omega)+\sup n_\gamma(\psi)}{2}]\rangle \mid \omega\in U_1 \& \psi\in U_2\}\cup\{\langle\langle\omega,\psi\rangle, [\frac{\inf m_\beta(\omega)+\inf m_\gamma(\psi)}{2}, \frac{\sup m_\beta(\omega)+\sup m_\gamma(\psi)}{2}], [\frac{\inf n_\beta(\omega)+\inf n_\gamma(\psi)}{2}, \frac{\sup n_\beta(\omega)+\sup n_\gamma(\psi)}{2}]\rangle \mid \omega\in U_1 \& \psi\in U_2\} =$$

$$\{\langle\langle\omega,\psi\rangle, [\max(\frac{\inf m_\alpha(\omega)+\inf m_\gamma(\psi)}{2}, \frac{\inf m_\beta(\omega)+\inf m_\gamma(\psi)}{2}), \max(\frac{\sup m_\alpha(\omega)+\sup m_\gamma(\psi)}{2}, \frac{\sup m_\beta(\omega)+\sup m_\gamma(\psi)}{2})], [\min(\frac{\inf n_\alpha(\omega)+\inf n_\gamma(\psi)}{2}, \frac{\inf n_\beta(\omega)+\inf n_\gamma(\psi)}{2}), \min(\frac{\sup n_\alpha(\omega)+\sup n_\gamma(\psi)}{2}, \frac{\sup n_\beta(\omega)+\sup n_\gamma(\psi)}{2})]\rangle \mid \omega\in U_1 \& \psi\in U_2\}.$$

Now we make the observation that

$$\max(\frac{\inf m_\alpha(\omega)+\inf m_\gamma(\psi)}{2}, \frac{\inf m_\beta(\omega)+\inf m_\gamma(\psi)}{2}) = \frac{\max(\inf m_\alpha(\omega), \inf m_\beta(\omega))+\inf m_\gamma(\psi)}{2} = \frac{\inf\max(m_\alpha(\omega), m_\beta(\omega))+\inf m_\gamma(\psi)}{2}.$$

$$\max(\frac{\sup m_\alpha(\omega)+\sup m_\gamma(\psi)}{2}, \frac{\sup m_\beta(\omega)+\sup m_\gamma(\psi)}{2}) =$$

$$\frac{\max(\sup m_\alpha(\omega), \sup m_\beta(\omega)) + \sup m_\gamma(\psi)}{2} = \frac{\sup \max(m_\alpha(\omega), m_\beta(\omega)) + \sup m_\gamma(\psi)}{2}.$$

$$\min(\frac{\inf n_\alpha(\omega) + \inf n_\gamma(\psi)}{2}, \frac{\inf n_\beta(\omega) + \inf n_\gamma(\psi)}{2}) = \frac{\min(\inf n_\alpha(\omega), \inf n_\beta(\omega)) + \inf n_\gamma(\psi)}{2} = \frac{\inf \min(n_\alpha(\omega), n_\beta(\omega)) + \inf n_\gamma(\psi)}{2}.$$

$$\min(\frac{\sup n_\alpha(\omega) + \sup n_\gamma(\psi)}{2}, \frac{\sup n_\beta(\omega) + \sup n_\gamma(\psi)}{2}) = \frac{\min(\sup n_\alpha(\omega), \sup n_\beta(\omega)) + \sup n_\gamma(\psi)}{2} = \frac{\sup \min(n_\alpha(\omega), n_\beta(\omega)) + \sup n_\gamma(\psi)}{2}.$$

Therefore,

$$(\alpha \cup \beta) \times_6 \gamma = (\alpha \times_6 \gamma) \cup (\beta \times_6 \gamma).$$

On the left side of (3), we have

$$(\alpha \cap \beta) \times_6 \gamma = \{\langle \omega, [\min(\inf m_\alpha(\omega), \inf m_\beta(\omega)), \min(\sup m_\alpha(\omega), \sup m_\beta(\omega))], [\max(\inf n_\alpha(\omega), \inf n_\beta(\omega)), \max(\sup n_\alpha(\omega), \sup n_\beta(\omega))]\rangle \mid \omega \in U\},$$

$$\times_6 \gamma = \{\langle\langle \omega, \psi\rangle, [\frac{\inf \min(m_\alpha(\omega), m_\beta(\omega)) + \inf m_\gamma(\psi)}{2}, \frac{\sup \min(m_\alpha(\omega), m_\beta(\omega)) + \sup m_\gamma(\psi)}{2}], [\frac{\inf \max(n_\alpha(\omega), n_\beta(\omega)) + \inf n_\gamma(\psi)}{2}, \frac{\sup \max(n_\alpha(\omega), n_\beta(\omega)) + \sup n_\gamma(\psi)}{2}]\rangle \mid \omega \in U_1 \& \psi \in U_2\}.$$

On the right side of (3), we have

$$(\alpha \times_6 \gamma) \cap (\beta \times_6 \gamma) = \{\langle\langle \omega, \psi\rangle, [\frac{\inf m_\alpha(\omega) + \inf m_\gamma(\psi)}{2}, \frac{\sup m_\alpha(\omega) + \sup m_\gamma(\psi)}{2}], [\frac{\inf n_\alpha(\omega) + \inf n_\gamma(\psi)}{2}, \frac{\sup n_\alpha(\omega) + \sup n_\gamma(\psi)}{2}]\rangle \mid \omega \in U_1 \& \psi \in U_2\} \cap \{\langle\langle \omega, \psi\rangle, [\frac{\inf m_\beta(\omega) + \inf m_\gamma(\psi)}{2}, \frac{\sup m_\beta(\omega) + \sup m_\gamma(\psi)}{2}], [\frac{\inf n_\beta(\omega) + \inf n_\gamma(\psi)}{2}, \frac{\sup n_\beta(\omega) + \sup n_\gamma(\psi)}{2}]\rangle \mid \omega \in U_1 \& \psi \in U_2\} =$$

$$\{\langle\langle \omega, \psi\rangle, [\min(\frac{\inf m_\alpha(\omega) + \inf m_\gamma(\psi)}{2}, \frac{\inf m_\beta(\omega) + \inf m_\gamma(\psi)}{2}), \min(\frac{\sup m_\alpha(\omega) + \sup m_\gamma(\psi)}{2}, \frac{\sup m_\beta(\omega) + \sup m_\gamma(\psi)}{2})], [\max(\frac{\inf n_\alpha(\omega) + \inf n_\gamma(\psi)}{2}, \frac{\inf n_\beta(\omega) + \inf n_\gamma(\psi)}{2}), \max(\frac{\sup n_\alpha(\omega) + \sup n_\gamma(\psi)}{2}, \frac{\sup n_\beta(\omega) + \sup n_\gamma(\psi)}{2})]\rangle \mid \omega \in U_1 \& \psi \in U_2\}.$$

Now we make the observation that

$$\min(\frac{\inf m_\alpha(\omega) + \inf m_\gamma(\psi)}{2}, \frac{\inf m_\beta(\omega) + \inf m_\gamma(\psi)}{2}) = \frac{\min(\inf m_\alpha(\omega), \inf m_\beta(\omega)) + \inf m_\gamma(\psi)}{2} = \frac{\inf \min(m_\alpha(\omega), m_\beta(\omega)) + \inf m_\gamma(\psi)}{2}.$$

$$\min(\frac{\sup m_\alpha(\omega) + \sup m_\gamma(\psi)}{2}, \frac{\sup m_\beta(\omega) + \sup m_\gamma(\psi)}{2}) = \frac{\min(\sup m_\alpha(\omega), \sup m_\beta(\omega)) + \sup m_\gamma(\psi)}{2} = \frac{\sup \min(m_\alpha(\omega), m_\beta(\omega)) + \sup m_\gamma(\psi)}{2}.$$

$$\max(\frac{\inf n_\alpha(\omega) + \inf n_\gamma(\psi)}{2}, \frac{\inf n_\beta(\omega) + \inf n_\gamma(\psi)}{2}) = \frac{\max(\inf n_\alpha(\omega), \inf n_\beta(\omega)) + \inf n_\gamma(\psi)}{2} = \frac{\inf \max(n_\alpha(\omega), n_\beta(\omega)) + \inf n_\gamma(\psi)}{2}.$$

$$\max(\frac{\sup n_\alpha(\omega) + \sup n_\gamma(\psi)}{2}, \frac{\sup n_\beta(\omega) + \sup n_\gamma(\psi)}{2}) = \frac{\max(\sup n_\alpha(\omega), \sup n_\beta(\omega)) + \sup n_\gamma(\psi)}{2} = \frac{\sup \max(n_\alpha(\omega), n_\beta(\omega)) + \sup n_\gamma(\psi)}{2}.$$

Therefore,

$$(\alpha \cap \beta) \times_6 \gamma = (\alpha \times_6 \gamma) \cap (\beta \times_6 \gamma). \qquad \square$$

Proposition 5 Given two universes U_1 and U_2. α and β are IVIFSs in U_1, C is an IVIFS in U_2. Then these equalities hold:

$$(\alpha + \beta) \times_6 \gamma \subset (\alpha \times_6 \gamma) + (\beta \times_6 \gamma) \tag{6}$$

$$(\alpha \cdot \beta) \times_6 \gamma \supset (\alpha \times_6 \gamma) \cdot (\beta \times_6 \gamma) \tag{7}$$

$$(\alpha @ \beta) \times_6 \gamma = (\alpha \times_6 \gamma) @ (\beta \times_6 \gamma) \tag{8}$$

$$\gamma \times_6 (\alpha + \beta) \subset (\gamma \times_6 \alpha) + (\gamma \times_6 \beta) \tag{9}$$

$$\gamma \times_6 (\alpha \cdot \beta) \supset (\gamma \times_6 \alpha) \cdot (\gamma \times_6 \beta) \tag{10}$$

$$\gamma \times_6 (\alpha @ \beta) = (\gamma \times_6 \alpha) @ (\gamma \times_6 \beta). \tag{11}$$

Next, we will give the detail of the process of proof for (6), (7) and (8). And (9), (10) and (11) are direct corollaries of (6), (7) and (8). First, we understand and observe the following lemma.

Lemma 1 [5] For all three numbers $\omega_1, \omega_2, \omega_3 \in [0, 1]$, the following inequality is valid:

$$\omega_3 \cdot (2 - \omega_1 - \omega_2 - \omega_3) + \omega_1 \cdot \omega_2 \geqslant 0.$$

Proof Here, let's assume $\hat{a} = \inf m_\alpha(\omega), \hat{b} = \sup m_\alpha(\omega), \hat{c} = \inf m_\beta(\omega), \hat{d} = \sup m_\beta(\omega), \hat{e} = \inf m_\gamma(\psi), \hat{f} = \sup m_\gamma(\psi), a_1 = \inf n_\alpha(\omega), b_1 = \sup n_\alpha(\omega), c_1 = \inf n_\beta(\omega), d_1 = \sup n_\beta(\omega), e_1 = \inf n_\gamma(\psi), f_1 = \sup n_\gamma(\psi)$.

On the left side of (6), we have

$$(\alpha+\beta) \times_6 \gamma = \{\langle \omega, [\hat{a}+\hat{c}-\hat{a}\cdot\hat{c}, \hat{b}+\hat{d}-\hat{b}\cdot\hat{d}],$$

$$[a_1\cdot c_1, b_1\cdot d_1]\rangle \mid \omega \in U\} \times_6 \gamma =$$

$$\{\langle\langle\omega,\psi\rangle, [\frac{\inf(\hat{a}+\hat{c}-\hat{a}\cdot\hat{c}, \hat{b}+\hat{d}-\hat{b}\cdot\hat{d})+\hat{e}}{2},$$

$$\frac{\sup(\hat{a}+\hat{c}-\hat{a}\cdot\hat{c}, \hat{b}+\hat{d}-\hat{b}\cdot\hat{d})+\hat{f}}{2}],$$

$$[\frac{\inf(a_1\cdot c_1, b_1\cdot d_1)+e_1}{2},$$

$$\frac{\sup(a_1\cdot c_1, b_1\cdot d_1)+f_1}{2}]\rangle \mid \omega \in U_1 \& \psi \in U_2\} =$$

$$\{\langle\langle\omega,\psi\rangle, [\frac{\hat{a}+\hat{c}-\hat{a}\cdot\hat{c}+\hat{e}}{2}, \frac{\hat{b}+\hat{d}-\hat{b}\cdot\hat{d}+\hat{f}}{2}],$$

$$[\frac{a_1\cdot c_1+e_1}{2}, \frac{b_1\cdot d_1+f_1}{2}]\rangle \mid \omega \in U_1 \& \psi \in U_2\}.$$

On the right side of (6), we have

$$(\alpha \times_6 \gamma)+(\beta \times_6 \gamma) = \{\langle\langle\omega,\psi\rangle, [\frac{\hat{a}+\hat{e}}{2}, \frac{\hat{b}+\hat{f}}{2}], [\frac{a_1+e_1}{2},$$

$$\frac{b_1+f_1}{2}]\rangle \mid \omega \in U_1 \& \psi \in U_2\} + \{\langle\langle\omega,\psi\rangle, [\frac{\hat{c}+\hat{e}}{2}, \frac{\hat{d}+\hat{f}}{2}],$$

$$[\frac{c_1+e_1}{2}, \frac{d_1+f_1}{2}]\rangle \mid \omega \in U_1 \& \psi \in U_2\} = \{\langle\langle\omega,\psi\rangle,$$

$$[\frac{\hat{a}+\hat{c}+2\hat{e}}{2} - \frac{(\hat{a}+\hat{e})\cdot(\hat{c}+\hat{e})}{4}, \frac{\hat{b}+\hat{d}+2\hat{f}}{2} -$$

$$\frac{(\hat{b}+\hat{f})\cdot(\hat{d}+\hat{f})}{4}], [\frac{(a_1+e_1)\cdot(c_1+e_1)}{4},$$

$$\frac{(b_1+f_1)\cdot(d_1+f_1)}{4}]\rangle \mid \omega \in U_1 \& \psi \in U_2\}.$$

According to Proposition 2 and Lemma 1, the following inequalities are obtained:

$$\frac{\hat{a}+\hat{c}-\hat{a}\cdot\hat{c}+\hat{e}}{2} - \frac{\hat{a}+\hat{c}+2\hat{e}}{2} + \frac{(\hat{a}+\hat{e})\cdot(\hat{c}+\hat{e})}{4} =$$

$$\frac{\hat{e}\cdot(\hat{a}+\hat{c}+\hat{e}-2)-\hat{a}\cdot\hat{c}}{4} \leqslant 0.$$

$$\frac{\hat{b}+\hat{d}-\hat{b}\cdot\hat{d}+\hat{f}}{2} - \frac{\hat{b}+\hat{d}+2\hat{f}}{2} + \frac{(\hat{b}+\hat{f})\cdot(\hat{d}+\hat{f})}{4} =$$

$$\frac{\hat{f}\cdot(\hat{b}+\hat{d}+\hat{f}-2)-\hat{b}\cdot\hat{d}}{4} \leqslant 0.$$

$$\frac{a_1\cdot c_1+e_1}{2} - \frac{(a_1+e_1)\cdot(c_1+e_1)}{4} =$$

$$\frac{a_1\cdot c_1+e_1\cdot(2-a_1-c_1-e_1)}{4} \geqslant 0.$$

$$\frac{b_1\cdot d_1+f_1}{2} - \frac{(b_1+f_1)\cdot(d_1+f_1)}{4} =$$

$$\frac{b_1\cdot d_1+f_1\cdot(2-b_1-d_1-f_1)}{4} \geqslant 0.$$

Therefore,

$$(\alpha+\beta) \times_6 \gamma \subset (\alpha \times_6 \gamma) + (\beta \times_6 \gamma).$$

We can get from the left of (7) that

$$(\alpha\cdot\beta) \times_6 \gamma = \{\langle\omega, [\hat{a}\cdot\hat{c}, \hat{b}\cdot\hat{d}], [a_1+c_1-a_1\cdot c_1, b_1+d_1-$$

$$b_1\cdot d_1]\rangle \mid \omega \in U\} \times_6 \gamma = \{\langle\langle\omega,\psi\rangle, [\frac{\hat{a}\cdot\hat{c}+\hat{e}}{2}, \frac{\hat{b}\cdot\hat{d}+\hat{f}}{2}],$$

$$[\frac{a_1+c_1-a_1\cdot c_1+e_1}{2}, \frac{b_1+d_1-b_1\cdot d_1+f_1}{2}]\rangle \mid \omega \in$$

$$U_1 \& \psi \in U_2\}.$$

On the right side of (7), we have

$$(\alpha \times_6 \gamma)\cdot(\beta \times_6 \gamma) = \{\langle\langle\omega,\psi\rangle, [\frac{\hat{a}+\hat{e}}{2}, \frac{\hat{b}+\hat{f}}{2}],$$

$$[\frac{a_1+e_1}{2}, \frac{b_1+f_1}{2}]\rangle \mid \omega \in U_1 \& \psi \in U_2\} \cdot \{\langle\langle\omega,\psi\rangle,$$

$$[\frac{\hat{c}+\hat{e}}{2}, \frac{\hat{d}+\hat{f}}{2}], [\frac{c_1+e_1}{2}, \frac{d_1+f_1}{2}]\rangle \mid \omega \in U_1 \& \psi \in U_2\} =$$

$$\{\langle\langle\omega,\psi\rangle, [\frac{(\hat{a}+\hat{e})\cdot(\hat{c}+\hat{e})}{4}, \frac{(\hat{b}+\hat{f})\cdot(\hat{d}+\hat{f})}{4}],$$

$$[\frac{a_1+c_1+2e_1}{2} - \frac{(a_1+e_1)\cdot(c_1+e_1)}{4}, \frac{b_1+d_1+2f_1}{2} -$$

$$\frac{(b_1+f_1)\cdot(d_1+f_1)}{4}]\rangle \mid \omega \in U_1 \& \psi \in U_2\}.$$

According to Proposition 2 and Lemma 1, the following inequalities are obtained:

$$\frac{(\hat{a}+\hat{e})\cdot(\hat{c}+\hat{e})}{4} - \frac{\hat{a}\cdot\hat{c}+\hat{e}}{2} =$$

$$\frac{-\hat{a}\cdot\hat{c}+\hat{e}(\hat{a}+\hat{c}+\hat{e}-2)}{4}\leqslant 0,$$

$$\frac{(\hat{b}+\hat{f})\cdot(\hat{d}+\hat{f})}{4}-\frac{\hat{b}\cdot\hat{d}+\hat{f}}{2}=$$

$$\frac{-\hat{b}\cdot\hat{d}+\hat{f}(\hat{b}+\hat{d}+\hat{f}-2)}{4}\leqslant 0,$$

$$\frac{a_1+c_1+2e_1}{2}-\frac{(a_1+e_1)\cdot(c_1+e_1)}{4}-$$

$$\frac{a_1+c_1-a_1\cdot c_1+e_1}{2}=$$

$$\frac{a_1\cdot c_1+e_1(2-a_1-c_1-e_1)}{4}\geqslant 0,$$

$$\frac{b_1+d_1+2f_1}{2}-\frac{(b_1+f_1)\cdot(d_1+f_1)}{4}-$$

$$\frac{b_1+d_1-b_1\cdot d_1+f_1}{2}=$$

$$\frac{b_1\cdot d_1+f_1(2-b_1-d_1-f_1)}{4}\geqslant 0.$$

Therefore,

$$(\alpha\cdot\beta)\times_6\gamma\supset(\alpha\times_6\gamma)\cdot(\beta\times_6\gamma).$$

On the left side of (8), we have

$$(\alpha@\beta)\times_6\gamma=$$

$$\{\langle\omega,[\frac{\hat{a}+\hat{c}}{2},\frac{\hat{b}+\hat{d}}{2}],[\frac{a_1+c_1}{2},\frac{b_1+d_1}{2}]\rangle\mid\omega\in U\}\times_6\gamma=$$

$$\{\langle\langle\omega,\psi\rangle,[\frac{(\hat{a}+\hat{c})/2+\hat{e}}{2},\frac{(\hat{b}+\hat{d})/2+\hat{f}}{2}],$$

$$[\frac{(a_1+c_1)/2+e_1}{2},\frac{(b_1+d_1)/2+f_1}{2}]\rangle\mid\omega\in$$

$$U_1\&\psi\in U_2\}=\{\langle\langle\omega,\psi\rangle,[\frac{\hat{a}+\hat{c}+2\hat{e}}{4},$$

$$\frac{\hat{b}+\hat{d}+2\hat{f}}{4}],[\frac{a_1+c_1+2e_1}{4},\frac{b_1+d_1+2f_1}{4}]\rangle$$

$$\mid\omega\in U_1\&\psi\in U_2\}=\{\langle\langle\omega,\psi\rangle,[\frac{\hat{a}+\hat{e}}{2},\frac{\hat{b}+\hat{f}}{2}],$$

$$[\frac{a_1+e_1}{2},\frac{b_1+f_1}{2}]\rangle\mid\omega\in U_1\&\psi\in U_2\}@\{\langle\langle\omega,\psi\rangle,$$

$$[\frac{\hat{c}+\hat{e}}{2},\frac{\hat{d}+\hat{f}}{2}],[\frac{c_1+e_1}{2},\frac{d_1+f_1}{2}]\rangle\mid\omega\in U_1\&$$

$$\psi\in U_2\}=(\alpha\times_6\gamma)@(\beta\times_6\gamma)$$

Therefore,

$$(\alpha@\beta)\times_6\gamma=(\alpha\times_6\gamma)@(\beta\times_6\gamma).\qquad\Box$$

Proposition 6 Given two universes U_1 and U_2. α and β are IVIFSs in U_1 and U_2, then two equalities hold:

$$\Box(\alpha\times_6\beta)=\Box\alpha\times_6\Box\beta \tag{12}$$

$$\Diamond(\alpha\times_6\beta)=\Diamond\alpha\times_6\Diamond\beta. \tag{13}$$

Proof On the left side of (12), we have

$$\Box(\alpha\times_6\beta)=\Box\{\langle\langle\omega,\psi\rangle,[\frac{\inf m_\alpha(\omega)+\inf m_\beta(\psi)}{2},$$

$$\frac{\sup m_\alpha(\omega)+\sup m_\beta(\psi)}{2}],[\frac{\inf n_\alpha(\omega)+\inf n_\beta(\psi)}{2},$$

$$\frac{\sup n_\alpha(\omega)+\sup n_\beta(\psi)}{2}]\rangle\mid\omega\in U_1\&\psi\in U_2\}=\{\langle\langle\omega,\psi\rangle,$$

$$[\frac{\inf m_\alpha(\omega)+\inf m_\beta(\psi)}{2},\frac{\sup m_\alpha(\omega)+\sup m_\beta(\psi)}{2}],$$

$$[\inf(\frac{\inf n_\alpha(\omega)+\inf n_\beta(\psi)}{2},\frac{\sup n_\alpha(\omega)+\sup n_\beta(\psi)}{2}),$$

$$1-\sup(\frac{\inf m_\alpha(\omega)+\inf m_\beta(\psi)}{2},$$

$$\frac{\sup m_\alpha(\omega)+\sup m_\beta(\psi)}{2})]\rangle\mid\omega\in U_1\&\psi\in U_2\}=$$

$$\{\langle\langle\omega,\psi\rangle,[\frac{\inf m_\alpha(\omega)+\inf m_\beta(\psi)}{2},$$

$$\frac{\sup m_\alpha(\omega)+\sup m_\beta(\psi)}{2}],[\frac{\inf n_\alpha(\omega)+\inf n_\beta(\psi)}{2},$$

$$1-\frac{\sup m_\alpha(\omega)+\sup m_\beta(\psi)}{2}]\rangle\mid\omega\in U_1\&\psi\in U_2\}.$$

And for the right side of (12), we have

$$\Box\alpha\times_6\Box\beta=\{\langle\omega,m_\alpha(\omega),[\inf n_\alpha(\omega),$$

$$1-\sup m_\alpha(\omega)]\rangle\mid\omega\in U_1\}\times_6\{\langle\psi,m_\beta(\psi),[\inf n_\beta(\psi),$$

$$1-\sup m_\beta(\psi)]\rangle\mid\psi\in U_2\}=\{\langle\langle\omega,\psi\rangle,$$

$$[\frac{\inf m_\alpha(\omega)+\inf m_\beta(\psi)}{2},\frac{\sup m_\alpha(\omega)+\sup m_\beta(\psi)}{2}],$$

$$[(\inf(\inf n_\alpha(\omega),1-\sup m_\alpha(\omega))+\inf(\inf n_\beta(\psi),$$

$$1-\sup m_\beta(\psi)))/2,(\sup(\inf n_\alpha(\omega),1-\sup m_\alpha(\omega))+$$

$$\sup(\inf n_\beta(\psi),1-\sup m_\beta(\psi)))/2]\rangle$$

$$\mid\omega\in U_1\&\psi\in U_2\}=\{\langle\langle\omega,\psi\rangle,$$

$$[\frac{\inf m_\alpha(\omega)+\inf m_\beta(\psi)}{2},\frac{\sup m_\alpha(\omega)+\sup m_\beta(\psi)}{2}],$$

$$[\frac{\inf n_\alpha(\omega)+\inf n_\beta(\psi)}{2},1-\frac{\sup m_\alpha(\omega)+\sup m_\beta(\psi)}{2}]\rangle\mid$$

$$\omega\in U_1\&\psi\in U_2\}.$$

Therefore,

$$\Box(\alpha\times_6\beta)=\Box\alpha\times_6\Box\beta.\qquad\Box$$

Proof On the left side of (13), we have

$$\Diamond(\alpha\times_6\beta)=\Diamond\{\langle\langle\omega,\psi\rangle,[\frac{\inf m_\alpha(\omega)+\inf m_\beta(\psi)}{2},$$

$$\frac{\sup m_\alpha(\omega)+\sup m_\beta(\psi)}{2}],[\frac{\inf n_\alpha(\omega)+\inf n_\beta(\psi)}{2},$$
$$\frac{\sup n_\alpha(\omega)+\sup n_\beta(\psi)}{2}]\rangle \mid \omega\in U_1\&\psi\in U_2\}=$$
$$\{\langle\langle\omega,\psi\rangle,[\inf(\frac{\inf m_\alpha(\omega)+\inf m_\beta(\psi)}{2},$$
$$\frac{\sup m_\alpha(\omega)+\sup m_\beta(\psi)}{2}),$$
$$1-\sup(\frac{\inf n_\alpha(\omega)+\inf n_\beta(\psi)}{2},$$
$$\frac{\sup n_\alpha(\omega)+\sup n_\beta(\psi)}{2})],$$
$$[\frac{\inf n_\alpha(\omega)+\inf n_\beta(\psi)}{2},\frac{\sup n_\alpha(\omega)+\sup n_\beta(\psi)}{2}]\rangle \mid \omega\in$$
$$U_1\&\psi\in U_2\}=$$
$$\{\langle\langle\omega,\psi\rangle,[\frac{\inf m_\alpha(\omega)+\inf m_\beta(\psi)}{2},$$
$$1-\frac{\sup n_\alpha(\omega)+\sup n_\alpha(\omega)}{2}],$$
$$[\frac{\inf n_\alpha(\omega)+\inf n_\beta(\psi)}{2},\frac{\sup n_\alpha(\omega)+\sup n_\beta(\psi)}{2}]\rangle \mid \omega\in$$
$$U_1\&\psi\in U_2\}.$$

On the right side of (13), we have

$$\Diamond\alpha\times_6\Diamond\beta=\{\langle\omega,[\inf m_\alpha(\omega),1-\sup n_\alpha(\omega)],n_\alpha(\omega)\rangle \mid$$
$$\omega\in U_1\}\times_6\{\langle\omega,[\inf m_\beta(\psi),1-\sup n_\beta(\psi)],n_\beta(\psi)\rangle \mid \omega\in$$
$$U_2\}=\{\langle\langle\omega,\psi\rangle,[(\inf(\inf m_\alpha(\omega),1-\sup n_\alpha(\omega))+$$
$$\inf(\inf m_\beta(\psi),1-\sup n_\beta(\psi)))/2,(\sup(\inf m_\alpha(\omega),$$
$$1-\sup n_\beta(\psi))+\sup(\inf m_\beta(\psi),1-\sup n_\beta(\psi)))/2],$$
$$[\frac{\inf n_\alpha(\omega)+\inf n_\beta(\psi)}{2},\frac{\sup n_\alpha(\omega)+\sup n_\beta(\psi)}{2}]\rangle \mid \omega\in$$
$$U_1\&\psi\in U_2\}=\{\langle\langle\omega,\psi\rangle,$$
$$[\frac{\inf m_\alpha(\omega)+\inf m_\beta(\psi)}{2},1-\frac{\sup n_\alpha(\omega)+\sup n_\alpha(\omega)}{2}],$$
$$[\frac{\inf n_\alpha(\omega)+\inf n_\beta(\psi)}{2},\frac{\sup n_\alpha(\omega)+\sup n_\beta(\psi)}{2}]\rangle \mid \omega\in$$
$$U_1\&\psi\in U_2\}.$$

Therefore,

$$\Diamond(\alpha\times_6\beta)=\Diamond\alpha\times_6\Diamond\beta. \qquad \square$$

Proposition 7 For all IVIFSs α and β,

$$*_i(\alpha\times_6\beta)=*_i\alpha\times_6 *_i\beta \tag{14}$$

where $i(1\leqslant i\leqslant 4)$ is a positive integer.

Proof On the left side of (14), we have

$$*_1(\alpha\times_6\beta)=*_1\{\langle\langle\omega,\psi\rangle,[\frac{\inf m_\alpha(\omega)+\inf m_\beta(\psi)}{2},$$
$$\frac{\sup m_\alpha(\omega)+\sup m_\beta(\psi)}{2}],[\frac{\inf n_\alpha(\omega)+\inf n_\beta(\psi)}{2},$$
$$\frac{\sup n_\alpha(\omega)+\sup n_\beta(\psi)}{2}]\rangle \mid \omega\in U_1\&\psi\in U_2\}=$$
$$\{\langle\langle\omega,\psi\rangle,[\inf(\frac{\inf m_\alpha(\omega)+\inf m_\beta(\psi)}{2},$$
$$\frac{\sup m_\alpha(\omega)+\sup m_\beta(\psi)}{2})],[(\inf(\frac{\inf n_\alpha(\omega)+\inf n_\beta(\psi)}{2},$$
$$\frac{\sup n_\alpha(\omega)+\sup n_\beta(\psi)}{2})]\rangle \mid \omega\in U_1\&\psi\in U_2\}=$$
$$\{\langle\langle\omega,\psi\rangle,[\frac{\inf m_\alpha(\omega)+\inf m_\beta(\psi)}{2}],$$
$$[\frac{\inf n_\alpha(\omega)+\inf n_\beta(\psi)}{2}]\rangle \mid \omega\in U_1\&\psi\in U_2\}.$$

And for the right side of (14), we have

$$*_1\alpha\times_6 *_1\beta=\{\langle\omega,\inf m_\alpha(\omega),\inf n_\alpha(\omega)\rangle \mid \omega\in U_1\}$$
$$\times_6\{\langle\psi,\inf m_\beta(\psi),\inf n_\beta(\psi)\rangle \mid \psi\in U_2\}=$$
$$\{\langle\langle\omega,\psi\rangle,[\frac{\inf\inf m_\alpha(\omega)+\inf\inf m_\beta(\psi)}{2},$$
$$\frac{\sup\inf m_\alpha(\omega)+\sup\inf m_\beta(\psi)}{2}],$$
$$[\frac{\inf\inf n_\alpha(\omega)+\inf\inf n_\beta(\psi)}{2},$$
$$\frac{\sup\inf n_\alpha(\omega)+\sup\inf n_\beta(\psi)}{2}]\rangle \mid \omega\in U_1\&\psi\in U_2\}=$$
$$\{\langle\langle\omega,\psi\rangle,[\frac{\inf m_\alpha(\omega)+\inf m_\beta(\psi)}{2}],$$
$$[\frac{\inf n_\alpha(\omega)+\inf n_\beta(\psi)}{2}]\rangle \mid \omega\in U_1\&\psi\in U_2\}.$$

For $i=2,3$ and 4 the proofs are analogous.
Therefore,

$$*_i(\alpha\times_6\beta)=*_i\alpha\times_6 *_i\beta. \qquad \square$$

4. Geometric interpretation

To be easier to understand and visualize, one geometric interpretation about the Cartesian product $\times_6$ over IVIFSs is presented in this paper. The geometric interpretation of the Cartesian product $\times_6$ over IFS is known as shown in Fig. 1 [3]. IVIFS is a generalization of IFS. Of course, the geometric interpretation of the Cartesian product of IVIFS can also be derived from the Cartesian product of IFS.

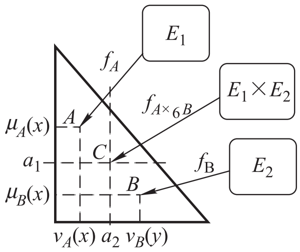


Fig. 1 Cartesian product $\times_6$ over IFSs

The theory of IFS relies on membership and non-membership functions, which are presented as follows: $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$, respectively.

A triangle structure of the Cartesian product of IFSs is provided in Fig. 1. We can build function $f_A : E_1 \longmapsto F$, $f_B : E_2 \longmapsto F$ and $f_{A\times_6 B} : E_1 \times E_2 \longmapsto F$ such that if $x \in E_1$ and $y \in E_2$ then $A = f_A(x) \in F; B = f_B(y) \in F; C = f_{A\times_6 B}(x,y) \in F$.

We can see three points from Fig. 1, namely A, B and C. They have the coordinates $(\mu_A(x), \nu_A(x))$, $(\mu_B(y), \nu_B(y))$ and $(\mu_{A\times_6 B}(x,y), \nu_{A\times_6 B}(x,y))$, respectively.

$$a_1 = \frac{\mu_A(x) + \mu_B(y)}{2}, \quad a_2 = \frac{\nu_A(x) + \nu_B(y)}{2}.$$

However, the membership function and the non-membership function of IVIFSs are $M_A : E \to INT[0,1]$ and $N_A : E \to INT[0,1]$, respectively. Similarly, let E_1 and E_2 denote two universes. A triangle structure of the Cartesian product of IVIFSs is provided in Fig. 2. We can build function $f_A : E_1 \longmapsto F$, $f_B : E_2 \longmapsto F$ and $f_{A\times_6 B} : E_1 \times E_2 \longmapsto F$. If $x \in E_1$ and $y \in E_2$, then the shaded quadrilateral $A_1A_2A_3A_4 = f_A(x) \in F$, the shaded quadrilateral $B_1B_2B_3B_4 = f_B(y) \in F$, and the heavily shaded quadrilateral $ABCD = f_{A\times_6 B}(x,y) \in F$.

$$a_3 = \frac{\inf M_A(x) + \inf M_B(y)}{2},$$

$$a_4 = \frac{\sup M_A(x) + \sup M_B(y)}{2},$$

$$a_5 = \frac{\inf N_A(x) + \inf N_B(y)}{2},$$

$$a_6 = \frac{\sup N_A(x) + \sup N_B(y)}{2}.$$

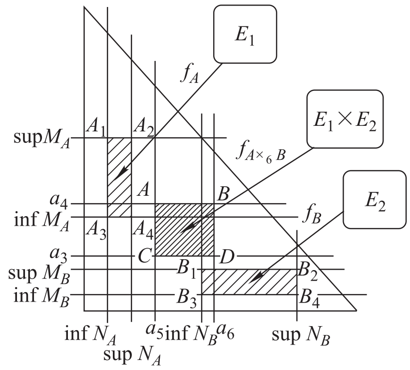


Fig. 2 Cartesian product $\times_6$ over IVIFSs

5. A numerical example

In Section 3, we propose the definition about the sixth Cartesian product of IVIFSs, then prove several relevant propositions. A specific numerical example is given to test the definition and proposition in this section. In other words, we assume that the membership and non-membership functions of IVIFSs are a specific interval. We then calculate and validate the proposed definitions and propositions over the sixth Cartesian product of IVIFSs. Let us consider the following IVIFSs A, B in $E_1 = \{1\}$ and $E_2 = \{2\}$, respectively, in order to look more intuitive and helpful to understand. Of course, the proposition is also valid for E_1 and E_2 are general universes. Let the IVIFSs A and B have the forms:

$$A = \{\langle 1, [0.1, 0.2], [0.3, 0.4]\rangle \mid 1 \in \{1\}\},$$

$$B = \{\langle 2, [0.2, 0.3], [0.4, 0.5]\rangle \mid 2 \in \{2\}\}$$

where $\sup M_A(x) + \sup N_A(x) = 0.6 \leqslant 1$ and $\sup M_B(x) + \sup N_B(x) = 0.8 \leqslant 1$.

Then we see that

$$A \times_6 B = \{\langle\langle 1,2\rangle, [\frac{0.1+0.2}{2}, \frac{0.2+0.3}{2}],$$

$$[\frac{0.3+0.4}{2}, \frac{0.4+0.5}{2}]\rangle \mid 1 \in \{1\} \& 2 \in \{2\}\}.$$

$$\overline{\overline{A} \times_6 \overline{B}} =$$

$$\overline{\{\langle\langle 1,2\rangle, [\frac{0.7}{2}, \frac{0.9}{2}], [\frac{0.3}{2}, \frac{0.5}{2}]\rangle \mid 1 \in \{1\} \& 2 \in \{2\}\}} =$$

$$\{\langle\langle 1,2\rangle, [\frac{0.3}{2}, \frac{0.5}{2}], [\frac{0.7}{2}, \frac{0.9}{2}]\rangle \mid 1 \in \{1\} \& 2 \in \{2\}\} =$$

$$A \times_6 B.$$

$$\Box(A \times_6 B) = \Box\{\langle\langle 1,2\rangle, [\frac{0.3}{2}, \frac{0.5}{2}], [\frac{0.7}{2}, \frac{0.9}{2}]\rangle \mid$$

$$1 \in \{1\} \& 2 \in \{2\}\} =$$

$$\{\langle\langle 1,2\rangle, [\frac{0.3}{2}, \frac{0.5}{2}], [\frac{0.7}{2}, \frac{1.5}{2}]\rangle \mid 1 \in \{1\} \& 2 \in \{2\}\}.$$

$$\square A \times_6 \square B = \{\langle 1, [0.1, 0.2], [0.3, 0.8]\rangle \mid 1 \in \{1\}\}$$

$$\times_6 \{\langle 2, [0.2, 0.3], [0.4, 0.7]\rangle \mid 2 \in \{2\}\} =$$

$$\{\langle\langle 1,2\rangle, [\frac{0.3}{2}, \frac{0.5}{2}], [\frac{0.7}{2}, \frac{1.5}{2}]\rangle \mid 1 \in \{1\} \& 2 \in \{2\}\} =$$

$$\square(A \times_6 B).$$

$$*_1(A \times_6 B) = *_1\{\langle\langle 1,2\rangle, [\frac{0.3}{2}, \frac{0.5}{2}], [\frac{0.7}{2}, \frac{0.9}{2}]\rangle \mid$$

$$1 \in \{1\} \& 2 \in \{2\}\} =$$

$$\{\langle\langle 1,2\rangle, \frac{0.3}{2}, \frac{0.7}{2}\rangle \mid 1 \in \{1\} \& 2 \in \{2\}\}.$$

$$*_1 A \times_6 *_1 B = \{\langle 1, 0.1, 0.3\rangle \mid 1 \in \{1\}\}$$

$$\times_6 \{\langle 2, 0.2, 0.4\rangle \mid 2 \in \{2\}\} =$$

$$\{\langle\langle 1,2\rangle, \frac{0.3}{2}, \frac{0.7}{2}\rangle \mid 1 \in \{1\} \& 2 \in \{2\}\} = *_1(A \times_6 B).$$

When the IVIFSs A, B in $E_1 = \{1\}$ and C in $E_2 = \{2\}$, let the IVIFSs A, B and C have the forms:

$$A = \{\langle 1, [0.1, 0.2], [0.3, 0.4]\rangle \mid 1 \in \{1\}\},$$

$$B = \{\langle 1, [0.2, 0.3], [0.4, 0.5]\rangle \mid 1 \in \{1\}\},$$

$$C = \{\langle 2, [0.3, 0.4], [0.5, 0.6]\rangle \mid 2 \in \{2\}\}.$$

In the same way, we can observe that

$$(A \cup B) \times_6 C = \{\langle 1, [0.2, 0.3], [0.3, 0.4]\rangle \mid$$

$$1 \in \{1\}\} \times_6 \{\langle 2, [0.3, 0.4], [0.5, 0.6]\rangle \mid 2 \in \{2\}\} =$$

$$\{\langle\langle 1,2\rangle, [\frac{0.2+0.3}{2}, \frac{0.3+0.4}{2}], [\frac{0.3+0.5}{2}, \frac{0.4+0.6}{2}]\rangle \mid$$

$$1 \in \{1\} \& 2 \in \{2\}\} =$$

$$\{\langle\langle 1,2\rangle, [\frac{0.5}{2}, \frac{0.7}{2}], [\frac{0.8}{2}, \frac{1}{2}]\rangle \mid 1 \in \{1\} \& 2 \in \{2\}\}.$$

$$(A \times_6 C) \cup (B \times_6 C) = \{\langle\langle 1,2\rangle, [\frac{0.4}{2}, \frac{0.6}{2}], [\frac{0.8}{2}, \frac{1}{2}]\rangle \mid$$

$$1 \in \{1\} \& 2 \in \{2\}\} \cup \{\langle\langle 1,2\rangle, [\frac{0.5}{2}, \frac{0.7}{2}], [\frac{0.9}{2}, \frac{1.1}{2}]\rangle \mid$$

$$1 \in \{1\} \& 2 \in \{2\}\} = \{\langle\langle 1,2\rangle, [\frac{0.5}{2}, \frac{0.7}{2}], [\frac{0.8}{2}, \frac{1}{2}]\rangle \mid$$

$$1 \in \{1\} \& 2 \in \{2\}\} = (A \cup B) \times_6 C.$$

$$(A + B) \times_6 C = \{\langle 1, [0.28, 0.44], [0.12, 0.2]\rangle \mid$$

$$1 \in \{1\}\} \times_6 \{\langle 2, [0.3, 0.4], [0.5, 0.6]\rangle \mid$$

$$2 \in \{2\}\} = \{\langle\langle 1,2\rangle, [\frac{0.58}{2}, \frac{0.88}{2}], [\frac{0.62}{2}, \frac{0.8}{2}]\rangle \mid$$

$$1 \in \{1\} \& 2 \in \{2\}\}.$$

$$(A \times_6 C) + (B \times_6 C) = \{\langle\langle 1,2\rangle, [\frac{0.4}{2}, \frac{0.6}{2}], [\frac{0.8}{2}, \frac{1}{2}]\rangle \mid$$

$$1 \in \{1\} \& 2 \in \{2\}\} + \{\langle\langle 1,2\rangle, [\frac{0.5}{2}, \frac{0.7}{2}], [\frac{0.9}{2}, \frac{1.1}{2}]\rangle \mid$$

$$1 \in \{1\} \& 2 \in \{2\}\} = \{\langle\langle 1,2\rangle, [\frac{0.8}{2}, \frac{1.09}{2}], [\frac{0.36}{2}, \frac{0.55}{2}]\rangle \mid$$

$$1 \in \{1\} \& 2 \in \{2\}\}.$$

It is obvious that $\frac{0.58}{2} < \frac{0.8}{2}, \frac{0.88}{2} < \frac{1.09}{2}, \frac{0.62}{2} > \frac{0.36}{2}, \frac{0.8}{2} > \frac{0.55}{2}$; therefore $(A+B) \times_6 C \subset (A \times_6 C) + (B \times_6 C)$.

It is also easy to notice that the other propositions are valid.

6. Conclusions

Based on the sixth Cartesian product of IFS and the five versions of Cartesian products over IVIFSs, the sixth Cartesian product of IVIFSs is presented in this paper. And some corresponding properties are proved. The difference between the sixth kinds of Cartesian products of IFSs and IVIFSs is: whether a membership function or a non-membership function, they must be a specific number or value, because they are specific in IFSs. However, they become an interval or range, because of the uncertainty in IVIFSs. We improve and perfect the type of Cartesian products of IVIFSs, which makes it match to the sixth Cartesian product of IFSs. Other Cartesian products of IVIFSs shall be studied in the future.

The properties of the sixth Cartesian product of IFS and IVIFSs are consistent. That is to say, the related properties of the Cartesian product of IFS can be extended to the IVIFSs. For example, the propositions 3, 4, 5 and 6 hold over IVIFS. In addition, the operators $*_i (i = 1, 2, 3, 4)$ satisfies the distribution rate of the sixth Cartesian product of IVIFS (proposition 7).

The geometric interpretation of the sixth Cartesian product of IFS is a point (the point C in Fig. 1). However, the geometric interpretation of the sixth Cartesian product of IVIFSs is an area (the quadrilateral $ABCD$ in Fig. 2). The reason is that the membership and non-membership functions of IVIFSs are an interval, respectively. Whether a membership function or a non-membership function in IFSs, it is a number or value.

Finally, this paper gives a numerical example to verify the proposition with some specific figures. We see that theory and practice are coincident. For the sake of simplicity, this numerical example has only one element in E_1 and E_2. Of course, suppose there are two or more elements in E_1 and E_2, the proposition is also valid.

References

[1] K. Atanassov. *Intuitionistic fuzzy sets: theory and applications* (*studies in fuzziness and soft computing*). Berlin Heidelberg: Springer-Verlag Telos, 1999.

[2] K. Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 1986, 20(1): 87 – 96.

[3] K. Atanassov, G. Gargov. Interval valued intuitionistic fuzzy sets. *Fuzzy sets and Systems*, 1989, 31(3): 343 – 349.

[4] D. S. Zeng, S. L. Hu. On index of interval valued intuitionistic fuzzy sets. *Journal of Sichuan Normal University* (*Natural Science*), 2000, 23(5): 476 – 478. (in Chinese)

[5] D. S. Zeng, Z. Z. Wu. Cartesian product and correlation of interval valued intuitionistic fuzzy sets. *Journal of Chengdu University of Information Technology*, 2004, 19(3): 458 – 462. (in Chinese)

[6] X. H. Yuan, H. X. Li. Cut sets on interval-valued intuitionistic fuzzy sets. *Proc. of the 6th IEEE International Conference on Fuzzy Systems and Knowledge Discovery*, 2009, 6: 167 – 171.

[7] X. H. Yuan, H. X. Li, K. B. Sun. Theory based on interval-valued level cut sets of Zadeh fuzzy sets. *Fuzzy Information and Engineering*, 2009, 2: 501 – 510.

[8] R. R. Yager. Some aspects of intuitionistic fuzzy sets. *Fuzzy Optimization and Decision Making*, 2009, 8(1): 67 – 90.

[9] Z. S. Xu. On similarity measures of interval-valued intuitionistic fuzzy sets and their application to pattern recognitions. *Journal of Southeast University* (*English Edition*), 2007, 23(1): 139 – 143.

[10] Q. S. Zhang, S. Y. Jiang, B. G. Jia, et al. Some information measures for interval-valued intuitionistic fuzzy sets. *Information Sciences*, 2010, 180(24): 5130 – 5145.

[11] Q. S. Zhang, H. X. Yao, Z. H. Zhang. Some similarity measures of interval-valued intuitionistic fuzzy sets and application to pattern recognition. *International Journal of Applied Mechanics and Materials*, 2010, 44 – 47(5): 495 – 500.

[12] G. W. Wei, J. M. Merig. Methods for strategic decision making problems with immediate probabilities in intuitionistic fuzzy setting. *Scientia Iranica E*, 2012, 19(6): 1936 – 1946.

[13] G. W. Wei, X. F. Zhao. Some induced correlated aggregating operators with intuitionistic fuzzy information and their application to multiple attribute group decision making. *Expert Systems with Applications*, 2012, 39(2): 2026 – 2034.

[14] Z. S. Xu, R. R. Yager. Intuitionistic and interval-valued intuitionistic fuzzy preference relations and their measures of similarity for the evaluation of agreement within a group. *Fuzzy Optimization and Decision Making*, 2009, 8(2): 123 – 139.

[15] D. F. Li. Topsis-based nonlinear-programming methodology for multiattribute decision making with Interval-valued intuitionistic fuzzy sets. *IEEE Trans. on Fuzzy Systems*, 2010, 18(2): 299 – 311.

[16] D. F. Li. Mathematical-programming approach to matrix games with payoffs represented by Atanassovs interval-valued intuitionistic fuzzy sets. *IEEE Trans. on Fuzzy Systems*, 2010, 18(6): 1112 – 1128.

[17] T. Y. Chen. IVIF-PROMETHEE outranking methods for multiple criteria decision analysis based on interval-valued intuitionistic fuzzy sets. *Fuzzy Optimization and Decision Making*, 2015, 14(2): 173 – 198.

[18] T. Y. Chen. The inclusion-based TOPSIS method with interval-valued intuitionistic fuzzy sets for multiple criteria group decision making. *Applied Soft Computing*, 2015, 26: 57 – 73.

[19] X. Y. Shen, Y. J. Lei, J. X. Hua, et al. Description and reasoning method of uncertain temporal knowledge based on IFTPN. *Control and Decision*, 2010, 25(10): 1457 – 1462.

[20] Q. S. Zhang, H. X. Yao, Z. H. Zhang. An interval-valued fuzzy reasoning approach based on weighted similarity measure. *Advanced Materials Research*, 2011, 143/144: 161 – 165.

[21] F. Y. Meng, X. H. Chen. Entropy and similarity measure for Atannasovs interval-valued intuitionistic fuzzy sets and their application. *Fuzzy Optimization and Decision Making*, 2016, 15(1): 75 – 101.

[22] P. Singh. A new similarity measure between interval-valued intuitionistic fuzzy sets. *Journal of Intelligent and Fuzzy Systems*. 2014, 27(1): 555 – 566.

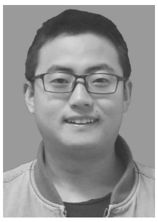
[23] A. Khalid, M. Abbas. Distance measures and operations in intuitionistic and interval-valued intuitionistic fuzzy soft set theory. *International Journal of Fuzzy Systems*, 2015, 17(3): 490 – 497.

[24] V. Andonov. On some properties of one Cartesian product over intuitionistic fuzzy sets. *Notes on Intuitionistic Fuzzy Sets*, 2008, 14(1): 12 – 19.

[25] A. Varghese, S. Kuriakose. Cartesian product over intuitionistic fuzzy sets. *International Journal of Fuzzy Sets*, 2012, 2(1): 21 – 27.

[26] A. Varghese, S. Kuriakose. More on Cartesian products over intuitionistic fuzzy sets. *International Mathematical Forum*, 2012, 7(23): 1129 – 1133.

Biographies



Jianming Xie was born in 1990. He received his B.S. degree in information and computing science from Shihezi University in 2015. Now he is a master at Xidian University. His research interests include fuzzy theory and its application.
E-mail: jmx726126@126.com

Sanyang Liu was born in 1959. He received his M.S. degree from Xidian University in 1984, and Ph.D. degree in Xi'an Jiaotong University in 1989. Now he is a famous professor of Xidian University. His research interest covers theory and application of optimization and network arithmetic.
E-mail: liusanyang@126.com