Improved grey prediction model based on exponential grey action quantity

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Abstract: With the passage of time, it has become important to investigate new methods for updating data to better fit the trends of the grey prediction model. The traditional GM(1,1) usually sets the grey action quantity as a constant. Therefore, it cannot effectively fit the dynamic characteristics of the sequence, which results in the grey model having a low precision. The linear grey action quantity model cannot represent the index change law. This paper presents a grey action quantity model, the exponential optimization grey model (EOGM(1,1)), based on the exponential type of grey action quantity; it is constructed based on the exponential characteristics of the grey prediction model. The model can fully reflect the exponential characteristics of the simulation series with time. The exponential sequence has a higher fitting accuracy. The optimized result is verified using a numerical example for the fluctuating sequence and a case study for the index of the tertiary industry's GDP. The results show that the model improves the precision of the grey forecasting model and reduces the prediction error.

Keywords: exponential of grey action quantity, optimal algorithm, grey forecasting, mathematical modeling.

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1. Introduction

Grey system theory (GST) was first proposed by Deng [1,2]. In recent years, GST has rapidly developed and has become an important research tool to solve the problem of modeling of small samples. The grey model GM(1,1) is a basic forecasting model of the GST. GM(1,1) has certain advantages such as fewer data requirements, simplicity, convenience, and accuracy. It is widely used in the fields of sociology [3], economics [4], engineering [5], transportation [6,7], and energy [8–10]. Liu [11] used the improved fractional grey prediction model to predict the emission of sulfur dioxide emissions in Beijing. Yuan [12]

used GM(1,1) and the autoregressive integrated moving average (ARIMA) model to predict the primary energy consumption in China and obtained future trends in China's energy consumption. Based on the combination of weight and GM(1,1) of the iron and steel enterprise, Li [13] provided useful advice on accident early warning systems for enterprise risk prevention and control.

The grey prediction model has been well established for more than 30 years now, and it has been elaborated and improved. The model is not limited to GM(1,1) with a single variable, such as the multivariable grey forecasting model [14–16]; it also includes the non-homogeneous exponential sequence grey prediction model [17,18], which greatly broadens the application range of the grey forecasting model. However, the GM(1,1) model, which is the basis of the grey prediction model, has much more room for improvement. The various improvements proposed for GM(1,1) have mainly focused on the following five points. First, the model optimizes the grey derivative. Wang [19] used a trapezoid formula to whiten the grey derivative, and then a Verhulst model definition was obtained to improve the uniformity of parameters. Wang [20] established a non-linear Bernoulli model to optimize the parameters in the differential equation and improve the prediction accuracy of the model. Li [21] proposed a method to optimize the grey derivative, derive the background value of the optimized GM(1,1), find the corresponding grey derivative, and optimize the model's precision. Mu [22,23] established an unbiased GM(1,1) by grey derivative optimization and proved that the model has a white exponential law coincidence property. Second, the model optimizes background values. Luo [24] obtained the optimized GM(1,1) background value by fitting the first-order accumulative generation sequence using a homogeneous exponential function. Xiong [25] used functions to fit the accumulated sequences for every variable and obtained the optimal formula for the background value of MGM (1, m) using a

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series of mathematical derivations. Xu [26] used the mean value theorem of integrals to study the relation between the background value and the development rate, making the background value unbiased and having the least error as compared to GM(1,1). To improve and optimize the model, Wang [27] proposed a grey prediction model based on the convolution integral corresponding to the background values of n variables. Third, the model optimizes the time response function. Liu [28] constructed the optimal model of the time response function GM(1,1) by finding the constant C in the time response sequence of the whiterization equation of GM(1,1). Cui [29] constructed an optimization model of the time response function for the non-equidistant GM(1,1) based on the condition that the sum of the residuals between the observations and the simulated values is small. Fourth, the model constructs a buffer operator. Cui [30,31] constructed a generalized weakening buffer operator and enhanced the buffer operator to solve the problem that the quantitative prediction results often do not correspond with the qualitative analysis conclusion in the process of modeling and forecasting of the shock disturbance data sequence problem. Wu [32] proposed the multivariable weakening buffer operator and demonstrated that the proposed method could effectively satisfy the new information condition of the sample; the buffer effect was obvious when the sample was small. Mao [33] proposed a grey model for fractional differential equations in GM(1,1), and the model followed the priority principle of new information. Fifth, Li [34] investigated the model properties and found that a constant that multiplies all data in the coarse series would influence the values of the model and the prediction. GM(1,1) was found to be suitable for a negative data sequence. Xiong [35] discussed the multiple transformations on the model parameters and the impact of the errors. The results have great significance for the further systematic study of MGM(1, m). The above-mentioned studies have brought about remarkable progress and have greatly promoted the development of the grey forecasting model. However, there is still limited research in the grey action quantity, which is an important part of the grey differential equation. Xu et al. [36] demonstrated that the grey action quantity has the property of changing with time. The grey function is used as a linear function of time for obtaining the dynamic grey action quantity, which is used to replace the parameter b , and the GM(1,1) differential equation is then obtained. However, this modification neglects the second-order Taylor expansion and the second-order residuals. The errors produced by the model will increase with time, and the structural relations of the variables will not be accurately represented. In this paper, we propose to further improve the grey action quantity b to a more precise form and obtain a better prediction of the simulation

results. As compared with the existing studies, this paper is unique in three aspects. First, we propose to optimize the grey action quantity in the grey differential equation. The parameter b in the grey action quantity of GM(1,1) is changed to the dynamic form with time. This enables us to avoid the errors with regards to second-order terms and the remainder of the Taylor expansion and improve the accuracy of the model. Secondly, we optimize the grey differential equation. This study presents the grey action quantity of the exponential form with time; therefore, the traditional grey differential equation will produce large errors in the calculation result. Based on this, the optimal form of the response function is obtained from the whitenization differential equation. Third, we estimate the parameters using the method of nonlinear least squares estimation. Owing to the existence of the exponential parameter in the grey differential equation, the linear least squares estimation is not applicable. In this paper, we have used the nonlinear least squares method to estimate the parameters of the differential equation; this is a rarity in previous studies. The rest of this paper is arranged as follows. In Section 2, the definition and theorem of GM(1,1) and GM(1,1) of the exponential grey action quantity optimization are introduced. The proof of the formula is given. The method of model verification is also described. In Section 3, we present a series of samples with exponential changes to validate GM(1,1) of the exponential optimization. GM(1,1) is used to optimize GM(1,1) and the improved model proposed in this paper; moreover, it is used to simulate the prediction and compare the results to determine the accuracy of the simulation and prediction. In Section 4, to further validate the effectiveness and applicability of the optimization model, a case of the index of the tertiary industry's GDP is used to compare the model. In Section 5, we draw our conclusions by comparing the various results obtained.

2. Modeling methods

2.1 Optimization model for GM(1,1)

In the traditional GM(1,1), the parameter b is used to represent the grey action quantity. For solving the model, b will be considered as a constant without any external interference. However, the grey effect in the practical study will change with time. Therefore, improving the grey effect when the error is accumulated in the grey prediction is critical.

Assuming that a set of original sample sequences is

$$X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \dots, x^{(0)}(n)). \quad (1)$$

The 1-accumulating generation operator (1-AGO) series can be obtained by imposing the first-order accumulating

generator operator:

$$X^{(1)} = (x^{(1)}(1), x^{(1)}(2), \dots, x^{(1)}(n)) \quad (2)$$

where $x^{(1)}(k) = \sum_{j=1}^k x^{(0)}(j)$, $k = 1, 2, \dots, n$.

We calculate the background value as

$$z^{(1)}(k) = \frac{1}{2}x^{(1)}(k) + \frac{1}{2}x^{(1)}(k-1), k = 2, 3, \dots, n. \quad (3)$$

From the above definition, GM(1,1) of the grey differential equation becomes

$$x^{(0)}(k) + az^{(1)}(k) = b. \quad (4)$$

Here, a is called the developing coefficient and b is called the grey action quantity. Therefore, the whitenization differential equation is written as

$$\frac{dx^{(1)}}{dt} + ax^{(1)} = b. \quad (5)$$

By solving the differential equation, we can obtain the time response sequence and then obtain the forecasted value as

$$\hat{x}^{(0)}(k+1) = (1-e^a)(X^{(0)}(1) - \frac{b}{a})e^{-ak}, k = 1, 2, \dots, n. \quad (6)$$

From (6), we can see that the forecasted value of the time response sequence is exponential with time. Li [34] proposed that the sample sequence changes to the original c times, and the number of developing coefficients of the sequence remains unchanged. The grey action quantity also changes c times; therefore, Xu [36] proposed the following theorem:

Theorem 1 [36] Suppose

$$X^{(0)}(k) = Ae^{\lambda(k-1)}, k = 1, 2, \dots$$

$$X_1^{(0)}(k) = Ae^{\lambda(k-1)}, k = 1, 2, \dots, N$$

$$X_2^{(0)}(k) = Ae^{\lambda(k-1)}, k = t+1, t+2, \dots, t+N.$$

$[a, b]$ is the parameter of $X_1^{(0)}(k)$, $[a', b']$ is the parameter of $X_2^{(0)}(k)$, therefore $a' = a$, $b' = e^{\lambda t}b$.

According to this theorem, the grey action quantity is regarded as a linear function with time. The linear GM(1,1) (hereinafter LGM (1,1)) with grey action quantity optimization is obtained by substituting $b_1 + b_2k$ for b .

The differential equation for LGM(1,1) is

$$x^{(0)}(k) + az^{(1)}(k) = b_1 + b_2k. \quad (7)$$

Correspondingly, the whitenization differential equation of LGM(1,1) is

$$\frac{dx^{(1)}}{dt} + ax^{(1)} = b_1 + b_2k. \quad (8)$$

If we assume that the least squares estimation parameter of (7) is $A = (a, b_1, b_2)^T$, then A satisfies

$$A = (B^T B)^{-1} B^T Y \quad (9)$$

$$\text{where } Y = \begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{bmatrix}, B = \begin{bmatrix} -z^{(1)}(2) & 1 & 2 \\ -z^{(1)}(3) & 1 & 3 \\ \vdots & \vdots & \vdots \\ -z^{(1)}(n) & 1 & n \end{bmatrix}.$$

Integrating the whitenization differential equation $\frac{dx^{(1)}}{dt} + ax^{(1)} = b_1 + b_2k$, we obtain the time response function as

$$\hat{x}^{(1)}(t) = (x^{(0)}(1) - \frac{b_1}{a} - \frac{b_2}{a} + \frac{b_2}{a^2})e^{-a(t-1)} + \frac{b_1}{a} + \frac{b_2}{a}t - \frac{b_2}{a^2}. \quad (10)$$

The above equation is discretized, and the time response of the grey differential equation $x^{(0)}(k) + az^{(1)}(k) = b_1 + b_2k$ is

$$\begin{aligned} \hat{x}^{(1)}(k+1) &= (x^{(0)}(1) - \frac{b_1}{a} - \frac{b_2}{a} + \frac{b_2}{a^2})e^{-ak} + \\ &\quad \frac{b_1}{a} + \frac{b_2}{a}(k+1) - \frac{b_2}{a^2}. \end{aligned} \quad (11)$$

The forecasted value is

$$\hat{x}^{(0)}(k+1) = (1-e^a)(x^{(0)}(1) - \frac{b_1}{a} - \frac{b_2}{a} + \frac{b_2}{a^2})e^{-ak} + \frac{b_2}{a}. \quad (12)$$

In Theorem 1, GM(1,1) is established when the time is changed from $k = 1$ to $k = t + 1$, and the grey action quantity is changed from b to $b' = e^{\lambda t}b$. GM(1,1) then takes a linear approximation, which is $e^{\lambda t}b \approx (1 + \lambda t)b$. This uses the Taylor formula at 0 as follows: $e^{\lambda t} = 1 + \lambda t + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + Rn(\lambda t)$. However, this ignores the second-order term, the remainder of elements in the Taylor formula, and the structural relations of variables. In this paper, we improve GM(1,1) by using the grey action quantity optimization, and propose an exponential optimization grey model (EOGM(1,1)).

2.2 Exponential grey action quantity optimization of GM(1,1)

The grey forecast model requires that the data satisfy the quasi-exponential rate. The exponential change of the sample data is an important part of the grey forecasting model. If we separate the exponential variation of the sample data sequence, we can see that the grey action quantity is time-dependent and that this change is exponential with time.

The traditional GM(1,1) treats the grey action quantity as a constant. Literature specifies the improvement as linearization, which is inconsistent with the changing trend of the grey action quantity. Therefore, the traditional method of dealing with the grey action quantity will inevitably affect the accuracy of the forecast, which would increase the estimated error. Therefore, this study considers the grey action quantity as an exponential function of time and proposes a GM(1,1) with grey exponential optimization.

Definition 1

$$\frac{dx^{(1)}}{dt} + ax^{(1)} = be^{-at} \quad (13)$$

is called the whitenization differential equation with exponential optimization.

Theorem 2

$$x^{(0)}(k) + az^{(1)}(k) = -\frac{b}{a}(1 - e^a)e^{-ak} \quad (14)$$

is called grey differential equation with exponential optimization.

Proof In the interval $[k-1, k]$, calculating the integral on both sides of the whitenization differential equation $\frac{dx^{(1)}}{dt} + ax^{(1)} = be^{-at}$, we obtain the following:

$$\int_{k-1}^k \frac{dx^{(1)}}{dt} dt + a \int_{k-1}^k x^{(1)} dt = \int_{k-1}^k be^{-at} dt. \quad (15)$$

Then,

$$\begin{aligned} & x^{(1)}(k) - x^{(1)}(k-1) + \\ & a \int_{k-1}^k x^{(1)} dt = -\frac{b}{a}(1 - e^a)e^{-ak}. \end{aligned} \quad (16)$$

By simplifying this, we obtain

$$x^{(0)}(k) + a \int_{k-1}^k x^{(1)}(t) dt = -\frac{b}{a}(1 - e^a)e^{-ak}. \quad (17)$$

When (18) is combined with the condition $\int_{k-1}^k x^{(1)}(t) dt = 0.5(x^{(1)}(k) + x^{(1)}(k-1)) = z^{(1)}(k)$, the grey differential equation with the exponential optimization becomes

$$x^{(0)}(k) + az^{(1)}(k) = -\frac{b}{a}(1 - e^a)e^{-ak}. \quad \square$$

For the parameters a and b in (13) and (14), it is difficult to obtain the analytical solution by the ordinary least squares method because of the existence of exponential parameters; therefore, the parameters cannot be estimated. Consequently, in this paper, we use the nonlinear least squares iterative algorithm to estimate the specific iterative

process using the computer implementation of the basic idea of the nonlinear least squares method.

The iteration algorithm can be performed as follows:

Step 1 Give the initial values a_0 and b_0 for the estimated parameter, and assign the number of iterative steps as $i = 1$.

Step 2 Determine a vector v as the iteration direction of step i .

Step 3 Determine the scalar step ρ using the optimal method.

Step 4 Set the iteration termination conditions. Check whether the termination conditions are met. If the conditions are not satisfied, then add 1 to i and repeat from Step 2. If the conditions are met, then take the parameter estimates as a_{i-1} and b_{i-1} .

The parameter estimation values $\hat{a}$ and $\hat{b}$ can be obtained using the iterative procedure of the above nonlinear least squares method. Substituting the estimated parameters into the whitenization equation and the grey differential equation, we can obtain the time response sequence and the forecasted value. Therefore, we obtain the following theorem.

Theorem 3 Let us suppose that the estimators of the constant are a and b , then we have the following:

(i) The time response function of the whitening equation $\frac{dx^{(1)}}{dt} + ax^{(1)} = be^{-ak}$ is

$$\hat{x}^{(1)}(t) = bte^{-at} + \frac{\sum_{i=1}^t (x^{(1)}(i)e^{-ai} - bie^{-2ai})}{\sum_{i=1}^t e^{-2ai}} e^{-at}. \quad (18)$$

(ii) The time response sequence of the grey differential equation $x^{(0)}(k) + az^{(1)}(k) = -\frac{b}{a}(1 - e^a)e^{-ak}$ is

$$\begin{aligned} \hat{x}^{(1)}(k) &= (bk + \frac{\sum_{i=1}^k (x^{(1)}(i)e^{-ai} - bie^{-2ai})}{\sum_{i=1}^k e^{-2ai}})e^{-ak}, \\ & k = 1, 2, \dots, n. \end{aligned} \quad (19)$$

(iii) The modeling and forecasting values are

$$\hat{x}^{(0)}(k) = (bke^{-a} + \frac{\sum_{i=1}^k (x^{(1)}(i)e^{-ai} - bie^{-2ai})}{\sum_{i=1}^k e^{-2ai}})e^{-a} -$$

$$b(k-1) - \frac{\sum_{i=1}^{k-1} (x^{(1)}(i)e^{-ai} - bie^{-2ai})}{\sum_{i=1}^{k-1} e^{-2ai}} e^{-a(k-1)},$$

$$k = 2, \dots, n. \quad (20)$$

Proof (i) The general solution of the differential equation $\frac{dx^{(1)}}{dt} + ax^{(1)} = be^{-ak}$ is

$$x^{(1)}(t) = e^{-\int a dt} \left[\int e^{-at} b e^{\int a dt} dt + c \right]. \quad (21)$$

Then,

$$x^{(1)}(t) = bte^{-at} + ce^{-at}. \quad (22)$$

Here, the parameter c is obtained by minimizing the sum of the squared differences of the first-order cumulant-generated sequence and the first-order cumulative-simulation sequence of samples.

First, the residual square sum function $E(c, k)$ of the first-order cumulant-generation sequence $X^{(1)}$ and its simulation sequence $\hat{X}^{(1)}$ is constructed as

$$E(c, k) = \sum_{i=1}^k (x^{(1)}(i) - \hat{x}^{(1)}(i))^2 =$$

$$\sum_{i=1}^k (x^{(1)}(i) - bie^{-ai} - ce^{-ai})^2. \quad (23)$$

This is combined with the condition $E'(c, k) = 0$; therefore,

$$c = \frac{\sum_{i=1}^k (x^{(1)}(i)e^{-ai} - bie^{-2ai})}{\sum_{i=1}^k e^{-2ai}}. \quad (24)$$

The time response function of the whitening equation $\frac{dx^{(1)}}{dt} + ax^{(1)} = be^{-ak}$ is

$$\hat{x}^{(1)}(t) = \left(bt + \frac{\sum_{i=1}^t (x^{(1)}(i)e^{-ai} - bie^{-2ai})}{\sum_{i=1}^t e^{-2ai}} \right) e^{-at}. \quad (25)$$

(ii) From (26), the time response of the grey differential equation is

$$\hat{x}^{(1)}(k) = \left(bk + \frac{\sum_{i=1}^k (x^{(1)}(i)e^{-ai} - bie^{-2ai})}{\sum_{i=1}^k e^{-2ai}} \right) e^{-ak},$$

$$k = 1, 2, \dots, n. \quad (26)$$

(iii) From (26), we have the following: $\hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1)$.

From (20), we can see that the estimated sequence is still exponential with time; therefore, the grey action quantity continues to be exponential. This confirms that the white-nization equation and the differential equation have improved. The improvement is an effective adjustment to the amount of grey action quantity. $\square$

2.3 Applicability analysis of the models

The grey prediction model is based on the prediction method established by the small sample and the uncertainty system. Although the application condition is more relaxed, the model has a certain scope of application, which is generally applicable to the change sequence with a fixed trend [37,38]. Based on the previous research, we construct the exponential change sequences, the linear oscillation sequences with a fixed trend and the sequences of exponential oscillation by changing the sequence of the sample. The paper studies the adaptability of the three models to the above three cases.

First construct the exponential change sequence $x_i^{(0)}(k) = e^{-ak}$, $-a = 0.1, 0.2, 0.4, 0.6, 1$ and $k = 0, 1, 2, 3, 4, 5$. Thus we can obtain sequences $X_1^{(0)}$, $X_2^{(0)}$, $X_3^{(0)}$, $X_4^{(0)}$, $X_5^{(0)}$. The above sequences are used as the original sequences to establish the models, and the average relative error of the sequences simulation are shown in the following table.

Table 1 Comparison of three models for the exponential change sequences

Sequence	$-a$	GM(1,1)/%	LGM(1,1)/%	EOGM(1,1)/%
$X_1^{(0)}$	0.1	0.105 2	0.119 6	0.115 1
$X_2^{(0)}$	0.2	0.498 3	0.493 2	0.362 1
$X_3^{(0)}$	0.4	2.613 4	2.614 0	0.810 0
$X_4^{(0)}$	0.6	8.182 8	5.493 9	1.364 8
$X_5^{(0)}$	1.0	23.543 7	23.547 2	1.955 1

From analysis of the above table we can see that when the development coefficient is small, the three models can accurately simulate the trend of sequence changes. With the increasing development coefficient, GM(1,1) and LGM(1,1) simulations are gradually distorted, but EOGM(1,1) can still be effectively estimated. Therefore, when the index changes gently, the three models can be effectively simulated. However, for the sequence with drastic changes in the index, the EOGM(1,1) model has a better adaptability.

Construct a linear oscillation sequence with a fixed trend $y_i^{(0)}(k) = a(k+1) + b$, $a = 0.5, 1, 2, 4, 10$ and $k = 0, 1, 2, 3, 4, 5$, where b is a random disturbance term, obtained by random values. In order to maintain the fixed

trend of the sequence, the range of b does not exceed $\frac{2}{3}a(k+1)$. The simulation results of the three models are shown in the following table.

Table 2 Comparison of three models for the linear oscillation sequences with fixed trend

Sequence	a	GM(1,1)/%	LGM(1,1)/%	EOGM(1,1)/%
$Y_1^{(0)}$	0.5	8.836 9	8.484 1	14.293 9
$Y_2^{(0)}$	1	7.563 9	8.575 2	5.268 9
$Y_3^{(0)}$	2	8.556 0	8.389 8	10.630 7
$Y_4^{(0)}$	4	10.028 0	15.588 4	13.116 6
$Y_5^{(0)}$	10	9.926 4	9.762 2	15.952 5

We can see that there are some errors in the three models of linear oscillation sequences with the fixed trend. There is no obvious advantage in the improved EOGM(1,1) model.

On the basis of constructing the exponential change sequence, we add the random disturbance term to construct the non-exponential change sequence. In order to maintain the fixed trend of the sequence, we set the range of the random disturbance term to no more than 50% of the original value. The simulation results of the three models are shown in the following table.

Table 3 Comparison of three models for the sequences of exponential oscillation

Sequence	$-a$	GM(1,1)/%	LGM(1,1)/%	EOGM(1,1)/%
$M_1^{(0)}$	0.1	4.297 6	5.521 5	3.469 3
$M_2^{(0)}$	0.2	10.579 8	13.102 6	7.394 1
$M_3^{(0)}$	0.4	3.952 7	7.874 9	2.378 6
$M_4^{(0)}$	0.6	14.498 3	13.610 8	9.959 2
$M_5^{(0)}$	1.0	29.909 4	34.977 6	4.320 1

From analysis of the table we can see that when the development coefficient is small, the three models can accurately simulate the trend of sequences changes, but the three models of the error have no increasing trend any longer. In terms of overall trend, GM(1,1) and LGM(1,1) simulations are gradually distorted with the increasing coefficient of development, and EOGM(1,1) can be effectively estimated. Therefore, for the exponential oscillation sequence, the EOGM(1,1) model has better adaptability.

Through the above analysis, we can find that the EOGM(1,1) simulation effect is superior to the other two models for exponential sequences and exponential oscillation sequences. While the three models have similar effects on linear oscillation sequences. Therefore, EOGM(1,1) is more suitable for simulations with a certain exponential trend or violent changes with a fixed trend.

2.4 Model verification

The relative percentage error (RPE) compares the real and forecasted values to evaluate the precision at a specific time

t . RPE is defined as

$$\varepsilon_t = \left| \frac{\hat{x}^{(0)}(t) - x^{(0)}(t)}{x^{(0)}(t)} \right| \times 100\%,$$

$$t = 2, 3, \dots, n. \quad (27)$$

The total model precision can be defined by the average relative percentage error (ARPE) as follows:

$$\bar{\varepsilon} = \frac{1}{n-1} \sum_{t=2}^n \varepsilon_t. \quad (28)$$

Given that $\alpha = 10\%$ when $\varepsilon_t < \alpha$ and $\bar{\varepsilon} < \alpha$, the model is a satisfactory residual model.

2.5 Procedure of the EOGM(1,1) model

Step 1 Let an original non-negative series be defined as

$$X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \dots, x^{(0)}(n)).$$

Then, the 1-AGO series of $X^{(0)}$ can be obtained by imposing the first-order accumulating generator operator $X^{(1)} = (x^{(1)}(1), x^{(1)}(2), \dots, x^{(1)}(n))$.

The background value series can be obtained as

$$z^{(1)}(k) = 0.5(x^{(1)}(k) + x^{(1)}(k-1)), k = 2, 3, \dots, n.$$

Step 2 Estimate the nonlinear least squares of a and b based on (14).

Step 3 Calculate the time response sequence of the grey differential equation based on (19).

$$\hat{x}^{(1)}(k) = bke^{-ak} + \frac{\sum_{i=1}^k (x^{(1)}(i)e^{-ai} - bie^{-2ai})}{\sum_{i=1}^k e^{-2ai}} e^{-ak},$$

$$k = 1, 2, \dots, n.$$

The forecasting values are obtained according to (20) as

$$\hat{x}^{(0)}(k+1) = (bk + x^{(0)}(1)e^a)e^{-a(k+1)} -$$

$$(b(k-1) + x^{(0)}(1)e^a)e^{-ak}.$$

Step 4 Use (28) to calculate the ARPE. According to the given error limit, determine whether the forecast results can be accepted. If it does not meet the conditions, consider optimizing the model or choose another model.

The above process can be used as shown in Fig. 1.

Fig. 1 EOGM flowchart

3. Numerical example of EOGM(1,1) model

Let us suppose that the raw data sequence is {100, 97.024 9, 95.084 1, 93.172 4, 91.289 3, 89.434 5, 87.607 5,

85.808 1, 84.026 9, 82.282 5}. First, the series requires a level test to verify that the model is feasible. The order of the adjacent numbers is {1.030 7, 1.020 4, 1.020 5, 1.020 6, 1.020 7, 1.020 9, 1.021 0, 1.021 2, 1.021 2}. All the numbers are in the permissible range $(e^{-\frac{2}{n+1}}, e^{\frac{2}{n+1}})$, which is (0.833 8, 1.199 4). Therefore, we can use the grey prediction model. The series shows a decreasing form; the decrease in difference gradually decreases, and the approximation appears to be exponentially decreasing. Using the exponential function fitting, we can see that the sequence approximates to $x^{(0)}(k) = 101.553 8e^{-0.0211k}$. GM(1,1), LGM(1,1), and EOGM(1,1) are established for the original sequence. In the modeling process, this paper uses the first eight numbers of the raw data sequence; the latter two numbers are used to forecast the effect of the prediction. Table 4 shows three models of simulated values and the RPEs. Table 5 presents the predicted value and the RPE.

Table 4 shows that the average absolute residuals of GM(1,1), LGM(1,1), and EOGM(1,1) are 0.025 6, 0.229 5, and 0.005 5, respectively. The average relative percentile errors for GM(1,1), LGM(1,1), and EOGM(1,1) are 0.028 0%, 0.251 4%, and 0.006 1%, respectively. At the same time, it is found that EOGM(1,1) is superior to the estimated value of the other two models for each simulation value and the RPE. The simulation results obtained from historical data demonstrate that the simulation accuracy of EOGM(1,1) is obviously higher than that of GM(1,1) and LGM(1,1); therefore, the model has been successfully improved.

Table 5 shows that the RPEs of the traditional GM(1,1) model for the one-step and two-step predictions are 0.024 0% and 0.064 4%, respectively. The RPEs of LGM(1,1) for the one-step and two-step predictions are 0.267 4% and 0.267 4%, respectively.

Table 4 Comparison of three models for simulated predictive value and RPE

Number	Actual value	GM(1,1) simulated value	LGM(1,1) simulated value	EOGM(1,1) simulated value	GM(1,1) -RPE/%	LGM(1,1) -RPE/%	EOGM(1,1) -RPE/%
1	100	100	100	100	0	0	0
2	97.024 9	97.067 2	97.264 9	97.016 7	0.043 6	0.247 4	0.008 5
3	95.084 1	95.090 4	95.320 5	95.079 6	0.006 6	0.248 6	0.004 7
4	93.172 4	93.153 9	93.405 2	93.166 8	0.019 9	0.249 9	0.006 0
5	91.289 3	91.256 8	91.518 7	91.284 0	0.035 6	0.251 3	0.005 8
6	89.434 5	89.398 4	89.660 5	89.430 1	0.040 4	0.252 7	0.004 9
7	87.607 5	87.577 8	87.830 2	87.601 9	0.033 9	0.254 2	0.006 4
8	85.808 1	85.794 3	86.027 4	85.802 9	0.016 1	0.255 6	0.006 1

Table 5 Comparison of predictive values and RPEs of the three models

Number	Actual value	GM(1,1) predictive value	LGM(1,1) predictive value	EOGM(1,1) predictive value	GM(1,1) -RPE/%	LGM(1,1) -RPE/%	EOGM(1,1) -RPE/%
9	84.026 9	84.047 1	84.251 6	84.025 8	0.024 0	0.267 4	0.001 3
10	82.282 5	82.335 5	82.502 5	82.280 5	0.064 4	0.267 4	0.002 4

The RPEs of EOGM(1,1) for the one-step and two-step predictions are 0.001 3% and 0.002 4%, respectively. A comparison of the predicted percentage errors of the three models shows that the predictive value of the EOGM(1,1) is closest to the actual value, which is better than the prediction effect of the other two models; it also has a higher practicability in the short- and medium-term forecast.

Fig. 2 represents the simulation prediction trends and deviations of the three models; the plotted graph predicts the percentage error. Using the above analysis, we can see that the EOGM(1,1) has a higher simulation prediction accuracy; moreover, it is more consistent with the development of raw data. In this paper, the EOGM(1,1) is used to simulate the domestic tertiary industry GDP index, and the validity and applicability of the optimization model are further verified.

Fig. 2 RPE histogram of the three models

4. Simulation and forecasting of the domestic tertiary industry GDP index

The tertiary industry refers to the service industry. Its developmental level is an important indicator of the degree

of socialization of production and the development of the market economy. It also includes the promotion of the market economic development, optimization of social resources (including natural resources, capital, and labor), improvement of the overall efficiency, and the efficiency of the national economy. To accelerate the development of China's tertiary industry, it is important to have an in-depth understanding of this industry. Therefore, we need to analyze the root causes of the various problems and examine the various laws and developmental trends. The tertiary industry GDP is an important indicator in the development of the tertiary industry. The method used to forecast the tertiary industry GDP is an important link in the development of the tertiary industry and developmental decisions. The GDP index of the tertiary industry refers to the relative trends and the degree of change of the tertiary industry during a certain period; it is calculated at constant prices and has a common trend with the tertiary industry's GDP.

The tertiary industry's GDP index forecasts both known information and unforeseen information, such as national policy, science and technology development, and international environment. Interference owing to these factors often makes it difficult to obtain long-term stable data; and the forecast system presents a grey feature. The tertiary industry's GDP index shows a monotonically increasing trend. The figures are non-negative; they are obtained annually, and the change ratio is controllable. This is in accordance with the grey model. The tertiary index is a grey system that dynamically changes based on various factors; therefore, it can be studied by using the grey system method. For this study, we have chosen the index of the tertiary industry in China from 2005 to 2014, as shown in Table 6. Based on the data from 2005 to 2012, GM(1,1), LGM(1,1), and EOGM(1,1) are established. The data from 2013 to 2014 are used to predict the two models.

Table 6 GDP index of the tertiary industry (Source: China Statistical Yearbook)

Year	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014
Index	1 590.7	1 815.3	2 106.8	2 327.1	2 547.1	2 794.1	3 059.7	3 303.3	3 576.0	3 856.6

The parameters of GM(1,1), LGM(1,1), EOGM(1,1) are estimated as

$$a = -0.094 6, b = 1 657.264 9,$$

$$a = -0.004 0, b_1 = 1 356.838 9, b_2 = 233.944 4,$$

$$a = -0.052 6, b = 1 617.732 6.$$

The time response sequences of GM(1,1), LGM(1,1), EOGM(1,1) are estimated as

$$\hat{x}^{(0)}(k+1) = 1 724.890 2e^{0.094 6k},$$

$$\hat{x}^{(0)}(k+1) = 60 131.345 4e^{0.004 0k} - 58 654.215 2,$$

$$\hat{x}^{(1)}(k) = 1 617.732 6ke^{-0.052 6k} +$$

$$\frac{\sum_{i=1}^k (x^{(1)}(i)e^{0.052 6i} - 1 617.732 6ie^{0.105 2i})}{\sum_{i=1}^k e^{0.105 2i}} e^{0.052 6t}.$$

Table 7 shows the simulated values and RPE for 2005–2012 for the three models. Table 8 shows the predicted values and the RPEs.

Table 7 Comparison of simulated values and relative percentage of tertiary industry in three models

Year	Actual value	GM(1,1) predictive value	LGM(1,1) predictive value	EOGM(1,1) predictive value	GM(1,1) -RPE/%	LGM(1,1) -RPE/%	EOGM(1,1) -RPE/%
2005	1 590.7	1 590.7	1 590.7	1 590.7	0	0	0
2006	1 815.3	1 896.7	1 717.4	1 847.3	4.451 1	5.393 0	1.762 8
2007	2 106.7	2 084.3	1 958.7	2 077.8	1.063 3	7.025 2	1.371 8
2008	2 327.1	2 291.1	2 201.0	2 302.6	1.547 0	5.418 8	1.052 8
2009	2 547.4	2 518.5	2 444.2	2 536.2	1.134 5	4.051 2	0.439 7
2010	2 794.1	2 768.5	2 688.3	2 786.9	0.916 2	3.786 6	0.257 7
2011	3 059.7	3 043.2	2 933.5	3 056.4	0.539 3	4.124 6	0.107 9
2012	3 303.3	3 345.3	3 179.6	3 333.8	1.271 5	3.744 7	0.923 3

Table 8 Forecast values and RPEs of GDP index of tertiary industry in 2013–2014

Year	Actual value	GM(1,1) predictive value	LGM(1,1) predictive value	EOGM(1,1) predictive value	GM(1,1) -ARPE/%	LGM(1,1) -ARPE/%	EOGM(1,1) -ARPE/%
2013	3 576.0	3 677.3	3 426.8	3 650.3	2.832 8	4.172 3	2.077 7
2014	3 856.6	4 042.3	3 674.9	3 987.4	4.815 1	4.711 4	3.391 6

Table 7 shows that the ARPE of GM(1,1), LGM(1,1), and EOGM(1,1) for the tertiary index are 1.560 4%, 4.792 0%, and 0.845 1%, respectively. EOGM(1,1) is found to have superior estimated values as compared to those of the other two models for each simulated value and RPE. It can be seen that EOGM(1,1) is more accurate than the other two models; therefore, the model improvement is deemed successful.

Table 8 shows that the RPEs of the GM(1,1) model for the one-step and two-step predictions are 2.832 8% and 4.815 1%, respectively. The RPEs of LGM(1,1) for the one-step and two-step predictions are 4.172 3% and 4.711 4%, respectively. The RPEs of EOGM(1,1) for the one-step and two-step predictions are 2.077 7% and 3.391 6%, respectively. By comparing the predicted percentage error of the three models, it can be found that the predictive value of the EOGM(1,1) is the closest to the actual value, and it has a higher practicability than the other models.

However, China is still a developing country, and several Chinese industries need further development. For a long time, the government has been paid more attention to the fields of science and technology, the service industry, the industrial structure, the domestic demand structure, and the tertiary industrial growth.

This paper is consistent with the grey prediction model proposed in this paper, which shows that the model can effectively reflect the tertiary index trends of the tertiary industry and conform to the actual social experience.

For a more intuitive representation of the three simulation models for predicting the trends of the domestic tertiary industry's GDP and the degree of deviation, the percentage errors of the three models are plotted in Fig. 3. By comparison, we can find that the three models are residual

qualified models. However, the accuracy of EOGM(1,1) is significantly higher than that of GM(1,1) and LGM(1,1). It has been shown that EOGM(1,1) has good simulation and short-term prediction effects; its simulation and prediction effect are better than the other two models. Therefore, the EOGM(1,1) proposed in this paper is effective and can compensate for the errors of the grey action quantity.

Fig. 3 RPE histogram of the three models

5. Conclusions

The classical grey prediction model is based on the exponential form of the data sequence. The grey action quantity in the differential equation changes with time. The traditional GM(1,1) usually sets the grey action quantity as a constant, which cannot effectively fit the dynamic characteristics of the data; this leads to a lower accuracy of the grey model. The existing grey action quantity optimization model cannot reflect the exponential change rule of

the data.

In this paper, we introduce the grey action quantity of the exponential time term and optimize the original GM(1,1) based on the characteristics of the grey prediction model simulation sequence. Based on the GM(1,1) for grey exponential optimization, a grey differential equation is proposed calculating the integral on both sides of the whitenization differential equation and the nonlinear least squares method to estimate the parameters. The exponential optimization GM(1,1) can fully reflect the exponential characteristics of the simulation sequence, and it has a higher fitting precision for the sequence satisfying the exponential form. To validate the model, GM(1,1), LGM(1,1), and EOGM(1,1) are used for modeling. The tertiary industry GDP index is simulated and predicted. The results show that all the three models can be simulated using residual tests, but EOGM(1,1) is more effective than the traditional GM(1,1) and LGM(1,1) models. The optimization model has a higher simulation and prediction accuracy. It has a certain theoretical value and application value.

In this study, we have improved the accuracy of the GM(1,1) model. However, the nonlinear least squares method is used to estimate the parameter values. The accuracy of different iterative algorithms is different, and there are some errors in the calculation, which lead to a deviation in the final result. Future studies can focus on improving the accuracy of the parameter estimation. For this study, the model used is based on the univariate exponential growth sequence. Future studies can explore how to extend this to the multivariable growth sequence and other forms of data sequences.

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