Linear complexity and autocorrelation of a new class of binary generalized cyclotomic sequences of order two and length pqr

JIA Wenjuan¹, GAO Juntao^{2,*}, and ZHANG Peng¹

1. School of Telecommunications Engineering, Xidian University, Xi'an 710071, China;

2. The State Key Laboratory of Integrated Services Network, Xidian University, Xi'an 710071, China

Abstract: Cyclotomic sequences have good cryptographic properties and are closely related to difference sets. This paper proposes a new class of binary generalized cyclotomic sequences of order two and length pqr . Its linear complexity, minimal polynomial, and autocorrelation are investigated. The results show that these sequences have a large linear complexity when $2 \in D_1$, which means they can resist the Berlekamp-Massey attack. Furthermore, the autocorrelation values are close to 0 with a probability of approximately $1 - 1/r$. Therefore, when r is a big prime, the new sequence has a good autocorrelation.

Keywords: generalized cyclotomic sequence, linear complexity, minimal polynomial, autocorrelation value.

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1. Introduction

The linear complexity of Legendre sequences was determined in [1,2]. In 1962, Whiteman introduced the generalized cyclotomy [3], where the motivation was to search for difference sets. Based on the Whiteman's generalized cyclotomy, Ding constructed generalized cyclotomic sequences of order two [4], and gave its linear complexity and autocorrelation in [5,6]. More works about linear complexities and autocorrelation values of Whiteman's generalized cyclotomic sequences can be referred to [7,8]. Later, a new generalized cyclotomy with respect to $p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ was introduced by Ding and Hellesteth [9].

Shortly after, based on Ding and Hellesteth's generalized cyclotomic class, the new generalized cyclotomic sequence of order two and order four were constructed and their pseudorandom properties were studied in [10–12]. Many researchers have focused on the linear complexity

and the autocorrelation values of generalized cyclotomic sequences with different characteristic sets, orders, and periods [13–23]. Moreover, some polyphase generalized cyclotomic sequences were studied in [24–26].

Chang et al. and Liu et al. constructed a new class of generalized cyclotomic sequences with respect to pqr over $GF(2)$, and determined its linear complexity with the cyclotomic theory. They showed that these sequences have a high linear complexity [27,28]. LYU et al. studied the linear complexity of a new generalized cyclotomic sequence of order two over $\mathbb{Z}_{p_1 p_2 p_3}$ [29].

This paper proposes a new class of binary generalized cyclotomic sequences of order two and the length $N = pqr$, and determines their linear complexities, minimal polynomials, and autocorrelation values. The new sequences have a different characteristic set from that in [27–29]. We show that the sequences have high linear complexity when $2 \in D_1$. Besides, we determine the minimal polynomial of the new sequences by calculating the great common divisor of $x^N - 1$ and $S(x)$. Moreover, the autocorrelation values are determined by considering properties of the quadratic character in finite fields.

Linear complexity and autocorrelation are very important indicators for measuring the security of sequences used in cryptography. The linear complexity $LC(z_i)$ of a T -periodic binary sequence (z_i) is defined as the length of the shortest linear feedback shift register (LFSR) which can generate the sequence. More specifically, the linear complexity is the smallest positive integer L satisfying

$$z_i = c_1 z_{i-1} + c_2 z_{i-2} + \cdots + c_L z_{i-L}$$

where $c_1, c_2, \dots, c_L \in GF(2)$ and $i \geq L$. The generating polynomial of (z_i) is defined by

$$Z(x) = z_0 + z_1 x + \cdots + z_{T-1} x^{T-1}.$$

It is well-known that the minimal polynomial of (z_i) is $m(x) = (x^T - 1) / \gcd(x^T - 1, Z(x))$, and the linear com-

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*Corresponding author.

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plexity of (z_i) is $LC(z_i) = T - \deg(\gcd(x^T - 1, Z(x)))$ (see [22, Lemma 8.2.1]). A low linear complexity (compared with $T/2$, where T denotes the period of the sequence) is not desirable for unpredictability of sequences in cryptography: if the linear complexity is l , then $2l$ consecutive bits of the sequence could be used to determine the whole sequence by using the Berlekamp-Massey algorithm.

The autocorrelation function of a binary sequence (z_i) with the period T is defined as

$$AC(z_i, \tau) = \frac{1}{T} \sum_{i \in \mathbb{Z}_T} (-1)^{z_i + z_{i+\tau}}$$

where $0 \leq \tau \leq T - 1$. Only when the values of $AC(z_i, \tau)$ distribute flat and low, the sequence (z_i) is easy to distinguish from each time shifted version of itself [18]. The autocorrelation function with the ideal distribution of values is two-valued, which is given by

$$AC(z_i, \tau) = \begin{cases} 1, & \tau = 0 \\ 1/T, & \text{otherwise} \end{cases}.$$

This paper is scheduled as follows. First we give the definition of new binary generalized cyclotomic sequences (s_i) in Section 2. In Section 3, we calculate the linear complexity and the minimal polynomial of (s_i) . The results show that the sequence has a high linear complexity when $2 \in D_1$. In Section 4, we study the autocorrelation values of (s_i) . In Section 5, we give the simulation and conclusion is made in Section 6.

2. Generalized cyclotomic sequences

Let p, q and r be odd primes with $d = \gcd(p-1, q-1) = \gcd(p-1, r-1) = \gcd(q-1, r-1) = 2$. By the Chinese remainder theorem, there exists a common primitive root g of p, q and r . Put $N = pqr, e = (p-1)(q-1)(r-1)/4$, and two positive integers x_1 and x_2 satisfy

$$\begin{cases} x_1 \equiv g \pmod{p} \\ x_1 \equiv 1 \pmod{q} \\ x_1 \equiv 1 \pmod{r} \end{cases} \begin{cases} x_2 \equiv 1 \pmod{p} \\ x_2 \equiv g \pmod{q} \\ x_2 \equiv 1 \pmod{r} \end{cases}.$$

Define

$$D_0 = \{g^{2s}, g^{2s}x_1, g^{2s}x_2, g^{2s}x_1x_2 : s = 0, 1, \dots, \frac{e}{2} - 1\}$$

$$D_1 = \{g^{2s+1}, g^{2s+1}x_1, g^{2s+1}x_2, g^{2s+1}x_1x_2 : s = 0, 1, \dots, \frac{e}{2} - 1\}.$$

We have $\mathbb{Z}_N^* = D_0 \cup D_1$ and $D_0 \cap D_1 = \emptyset$, where $\emptyset$ denotes the empty set. Define

$$P = \{p, 2p, \dots, (qr-1)p\}$$

$$P_1 = \{pq, 2pq, \dots, (r-1)pq\}$$

$$P_2 = \{pr, 2pr, \dots, (q-1)pr\}$$

$$P_3 = P - P_1 - P_2.$$

Furthermore, we divide P_1, P_2 and P_3 into two subsets respectively, that is

$$D_0^{(r)} = \{g^{2s} \bmod r : s = 0, 1, \dots, \frac{r-1}{2} - 1\}$$

$$D_1^{(r)} = \{g^{2s+1} \bmod r : s = 0, 1, \dots, \frac{r-1}{2} - 1\}$$

$$P_{10} = pqD_0^{(r)}, \quad P_{11} = pqD_1^{(r)}, \quad P_1 = P_{10} \cup P_{11}$$

$$D_0^{(q)} = \{g^{2s} \bmod q : s = 0, 1, \dots, \frac{q-1}{2} - 1\}$$

$$D_1^{(q)} = \{g^{2s+1} \bmod q : s = 0, 1, \dots, \frac{q-1}{2} - 1\}$$

$$P_{20} = prD_0^{(q)}, \quad P_{21} = prD_1^{(q)}, \quad P_2 = P_{20} \cup P_{21}$$

$$D_0^{(qr)} = \{g^{2s} \bmod qr, g^{2s}x \bmod qr : s = 0, 1, \dots, \frac{e_{qr}}{2} - 1\}$$

$$D_1^{(qr)} = \{g^{2s+1} \bmod qr, g^{2s+1}x \bmod qr : s = 0, 1, \dots, \frac{e_{qr}}{2} - 1\}$$

$$P_{30} = pD_0^{(qr)}, \quad P_{31} = pD_1^{(qr)}, \quad P_3 = P_{30} \cup P_{31}$$

where $e_{qr} = \frac{(q-1)(r-1)}{2}$, $x \equiv g \pmod{q}$ and $x \equiv 1 \pmod{r}$, then

$$\mathbb{Z}_r^* = D_0^{(r)} \cup D_1^{(r)}, \quad D_0^{(r)} \cap D_1^{(r)} = \emptyset$$

$$\mathbb{Z}_q^* = D_0^{(q)} \cup D_1^{(q)}, \quad D_0^{(q)} \cap D_1^{(q)} = \emptyset$$

$$\mathbb{Z}_{qr}^* = D_0^{(qr)} \cup D_1^{(qr)}, \quad D_0^{(qr)} \cap D_1^{(qr)} = \emptyset.$$

Define

$$Q = \{q, 2q, \dots, (pr-1)q\}$$

$$Q_1 = \{pq, 2pq, \dots, (r-1)pq\} = P_1$$

$$Q_2 = \{qr, 2qr, \dots, (p-1)qr\}$$

$$Q_3 = Q - Q_1 - Q_2$$

$$R = \{r, 2r, \dots, (pq-1)r\} - P_2 - Q_2.$$

Note that we divide Q_2, Q_3 and R into two subsets respectively, that is,

$$D_0^{(p)} = \{g^{2s} \bmod p : s = 0, 1, \dots, \frac{p-1}{2} - 1\}$$

$$D_1^{(p)} = \{g^{2s+1} \bmod p : s = 0, 1, \dots, \frac{p-1}{2} - 1\}$$

$$Q_{20} = qrD_0^{(p)}, \quad Q_{21} = qrD_1^{(p)}, \quad Q_2 = Q_{20} \cup Q_{21}$$

$$D_0^{(pr)} = \{g^{2s} \bmod pr, g^{2s}y \bmod pr : s = 0, 1, \dots, \frac{e_{pr}}{2} - 1\}$$

$$D_1^{(pr)} = \{g^{2s+1} \bmod pr, g^{2s+1}y \bmod pr : s = 0, 1, \dots, \frac{e_{pr}}{2} - 1\}$$

$$Q_{30} = qD_0^{(pr)}, \quad Q_{31} = qD_1^{(pr)}, \quad Q_3 = Q_{30} \cup Q_{31}$$

$$D_0^{(pq)} = \{g^{2s} \bmod pq, g^{2s}z \bmod pq : s = 0, 1, \dots, \frac{e_{pq}}{2} - 1\}$$

$$D_1^{(pq)} = \{g^{2s+1} \bmod pq, g^{2s+1}z \bmod pq : s = 0, 1, \dots, \frac{e_{pq}}{2} - 1\}$$

$$R_0 = rD_0^{(pq)}, \quad R_1 = rD_1^{(pq)}, \quad R = R_0 \cup R_1$$

$$O = \{0\}$$

where $e_{pr} = \frac{(p-1)(r-1)}{2}$, $y \equiv g \pmod{p}$ and $y \equiv 1 \pmod{r}$, $e_{pq} = \frac{(p-1)(q-1)}{2}$, $z \equiv g \pmod{p}$ and $z \equiv 1 \pmod{q}$, then

$$\mathbb{Z}_p^* = D_0^{(p)} \cup D_1^{(p)}, \quad D_0^{(p)} \cap D_1^{(p)} = \emptyset$$

$$\mathbb{Z}_{pr}^* = D_0^{(pr)} \cup D_1^{(pr)}, \quad D_0^{(pr)} \cap D_1^{(pr)} = \emptyset$$

$$\mathbb{Z}_{pq}^* = D_0^{(pq)} \cup D_1^{(pq)}, \quad D_0^{(pq)} \cap D_1^{(pq)} = \emptyset.$$

We have $\mathbb{Z}_N = D_0 \cup D_1 \cup P \cup Q_2 \cup Q_3 \cup R \cup O$. Define $C_0 = D_0 \cup Q_3 \cup R \cup O$, $C_1 = D_1 \cup P \cup Q_2$.

Based on the generalized cyclotomy, the new binary generalized cyclotomic sequence (s_i) of order two and the length pqr over $GF(2)$ is defined by

$$s_i = \begin{cases} 0, & i \bmod N \in C_0 \\ 1, & i \bmod N \in C_1 \end{cases} \quad (1)$$

where $i \geq 0$.

3. Linear complexity and minimal polynomial

The generating polynomial of (s_i) is defined as $S(x) = \sum_{i \in C_1} x^i = \sum_{i \in D_1} x^i + \sum_{i \in P} x^i + \sum_{i \in Q_2} x^i \in GF(2)[x]$. Let α be a primitive N th root of unit over the field $GF(2^m)$, where $m = \text{ord}_N(2)$, we have

$$LC(s_i) = N - |\{j : S(\alpha^j) = 0, 0 \leq j \leq N-1\}|. \quad (2)$$

If $x = \alpha^0 = 1$, we have

$$S(1) = \sum_{i \in D_1} 1 + \sum_{i \in P} 1 + \sum_{i \in Q_2} 1 \equiv 0 \pmod{2}. \quad (3)$$

Lemma 1 [28] Over the ring of $GF(2)[x]$, $\sum_{i \in \mathbb{Z}_N} \alpha^i =$

$$\sum_{i \in P} \alpha^i = \sum_{i \in Q} \alpha^i = \sum_{i \in R} \alpha^i = 1, \quad \sum_{i \in P_1} \alpha^i = \sum_{i \in P_2} \alpha^i = \sum_{i \in P_3} \alpha^i = 1, \quad \sum_{i \in Q_2} \alpha^i = \sum_{i \in Q_3} \alpha^i = 1.$$

Lemma 2 [30] If $a \in D_j$, then $aD_i = D_{(i+j) \pmod{2}}$ ($i, j = 0, 1$).

Lemma 3 Let the symbols be the same as before, then

$$S(\alpha^k) = \begin{cases} S(\alpha), & k \in D_0 \\ 1 + S(\alpha), & k \in D_1 \\ 1, & k \in P \cup Q_2 \\ 0, & k \in O \cup Q_3 \\ \frac{r-1}{2} \pmod{2}, & k \in R \end{cases}.$$

Proof If $k \in D_0$, by Lemma 2, $kP = P, kQ_2 = Q_2$ and $kD_1 = D_1$, then

$$S(\alpha^k) = \sum_{i \in D_1} \alpha^{ki} + \sum_{i \in P} \alpha^{ki} + \sum_{i \in Q_2} \alpha^{ki} = \sum_{i \in D_1} \alpha^i + \sum_{i \in P} \alpha^i + \sum_{i \in Q_2} \alpha^i = S(\alpha).$$

If $k \in D_1$, by Lemma 2, $kP = P, kQ_2 = Q_2$ and $kD_1 = D_0$, then

$$S(\alpha^k) = \sum_{i \in D_1} \alpha^{ki} + \sum_{i \in P} \alpha^{ki} + \sum_{i \in Q_2} \alpha^{ki} = \sum_{i \in D_0} \alpha^i + \sum_{i \in P} \alpha^i + \sum_{i \in Q_2} \alpha^i = 1 + S(\alpha).$$

(i) If $k \in P_1$, we get

$$D_1 \bmod r = \{g^{2s+1} \bmod r : s = 0, 1, \dots, \frac{e}{2} - 1\} = D_1^{(r)},$$

when s ranges over $\{0, 1, \dots, \frac{e}{2} - 1\}$, and $D_1 \bmod r$ runs through $D_1^{(r)}(p-1)(q-1)$ times, then

$$\sum_{i \in D_1} \alpha^{ki} = ((p-1)(q-1) \bmod 2) \sum_{i \in P_{11}} \alpha^i = 0.$$

Since $kP = P_1 \cup O$, we have

$$\sum_{i \in P} \alpha^{ki} = (q \bmod 2) \sum_{i \in P_1} \alpha^i + (q-1) \bmod 2 = 1.$$

Since $kQ_2 = O$, we have

$$\sum_{i \in Q_2} \alpha^{ki} = (p-1) \bmod 2 = 0.$$

Then

$$S(\alpha^k) = \sum_{i \in D_1} \alpha^{ki} + \sum_{i \in P} \alpha^{ki} + \sum_{i \in Q_2} \alpha^{ki} = 1.$$

(ii) If $k \in P_2$, we get

$$D_1 \bmod q = \{g^{2s} \bmod q, g^{2s+1} \bmod q : s =$$

$$0, 1, \dots, \frac{e}{2} - 1\} = \mathbb{Z}_q^*,$$

when s ranges over $\{0, 1, \dots, \frac{e}{2} - 1\}$, and $D_1 \bmod q$ runs through $\mathbb{Z}_q^* \frac{(p-1)(r-1)}{2}$ times, then

$$\sum_{i \in D_1} \alpha^{ki} = \left(\frac{(p-1)(r-1)}{2} \bmod 2 \right) \sum_{i \in P_2} \alpha^i = 0.$$

Since $kP = P_2 \cup O$, we have

$$\sum_{i \in P} \alpha^{ki} = (r \bmod 2) \sum_{i \in P_2} \alpha^i + (r-1) \bmod 2 = 1.$$

Since $kQ_2 = O$, we have

$$\sum_{i \in Q_2} \alpha^{ki} = (p-1) \bmod 2 = 0.$$

Then

$$S(\alpha^k) = \sum_{i \in D_1} \alpha^{ki} + \sum_{i \in P} \alpha^{ki} + \sum_{i \in Q_2} \alpha^{ki} = 1.$$

(iii) If $k \in P_3$, we get

$$D_1 \bmod qr = \{g^{2s+1} \bmod qr, g^{2s+1} x_2 \bmod qr : s =$$

$$0, 1, \dots, \frac{e}{2} - 1\} = D_1^{(qr)},$$

when s ranges over $\{0, 1, \dots, \frac{e}{2} - 1\}$, and $D_1 \bmod qr$ runs through $D_1^{(qr)} p - 1$ times, then

$$\sum_{i \in D_1} \alpha^{ki} = ((p-1) \bmod 2) \sum_{i \in P_{31}} \alpha^i = 0.$$

Since $kP = P$, we have

$$\sum_{i \in P} \alpha^{ki} = \sum_{i \in P} \alpha^i = 1.$$

Since $kQ_2 = O$, we have

$$\sum_{i \in Q_2} \alpha^{ki} = (p-1) \bmod 2 = 0.$$

Then

$$S(\alpha^k) = \sum_{i \in D_1} \alpha^{ki} + \sum_{i \in P} \alpha^{ki} + \sum_{i \in Q_2} \alpha^{ki} = 1.$$

From (i), (ii) and (iii), we have $S(\alpha^k) = 1$ when $k \in P$.

Similarly, we have $S(\alpha^k) = 1$ when $k \in Q_2$, and $S(\alpha^k) = 1$ when $k \in Q_3$.

(iv) If $k \in O$, we have

$$S(\alpha^k) = S(1) = 0.$$

(v) If $k \in R$, we have

$$D_1 \bmod pq = \mathbb{Z}_{pq}^*,$$

when s ranges over $\{0, 1, \dots, \frac{e}{2} - 1\}$, and $D_1 \bmod pq$ runs through $\mathbb{Z}_{pq}^* \frac{r-1}{2}$ times, then

$$\sum_{i \in D_1} \alpha^{ki} = \left(\frac{r-1}{2} \bmod 2 \right) \sum_{i \in R} \alpha^i = \frac{r-1}{2} \bmod 2.$$

Since $kP = P_2 \cup O$, we have

$$\sum_{i \in P} \alpha^{ki} = (r \bmod 2) \sum_{i \in P_2} \alpha^i + (r-1) \bmod 2 = 1.$$

Since $kQ_2 = Q_2$, we have

$$\sum_{i \in Q_2} \alpha^{ki} = \sum_{i \in Q_2} \alpha^i = 1.$$

Then

$$S(\alpha^k) = \sum_{i \in D_1} \alpha^{ki} + \sum_{i \in P} \alpha^{ki} + \sum_{i \in Q_2} \alpha^{ki} = \frac{r-1}{2} \bmod 2.$$

□

Lemma 4 $S(\alpha) \in \{0, 1\}$ if and only if $2 \in D_0$.

Proof Since the characteristic of the finite field $GF(2^m)$ is 2, it follows that $S^2(\alpha) = S(\alpha^2)$. By Lemma 3,

we have $S(\alpha^2) = S(\alpha)$ if and only if $2 \in D_0$. □

Theorem 1 Let notations be the same as before, then the linear complexity $LC(s_i)$ of the binary generalized cyclotomic sequence (s_i) satisfies

(i) If $r \equiv 1 \bmod 4$,

$$LC(s_i) = \begin{cases} pqr - \frac{(p-1)(q-1)(r-1)}{2} - \delta, & 2 \in D_0 \\ pqr - \delta, & 2 \in D_1 \end{cases}$$

where $\delta = (p-1)(q-1) + (p-1)(r-1) + 1$.

(ii) If $r \equiv 3 \bmod 4$,

$$LC(s_i) = \begin{cases} pqr - \frac{(p-1)(q-1)(r-1)}{2} - \eta, & 2 \in D_0 \\ pqr - \eta, & 2 \in D_1 \end{cases}$$

where $\eta = (p-1)(r-1) + 1$.

Proof (i) If $r \equiv 1 \bmod 4$, and $2 \in D_0$, by Lemma 3, only one of $S(\alpha)$ and $1 + S(\alpha)$ equals 0, here we choose α such that $S(\alpha) = 0$. Thus,

$$S(\alpha^k) = \begin{cases} 0, & k \in D_0 \\ 1, & k \in D_1 \\ 1, & k \in P \cup Q_2 \\ 0, & k \in O \cup Q_3 \cup R \end{cases}.$$

Combined with (2), we get

$$LC(s_i) = pqr - \frac{(p-1)(q-1)(r-1)}{2} - 1 - \\ (p-1)(r-1) - (p-1)(q-1) = \\ pqr - \frac{(p-1)(q-1)(r-1)}{2} - \delta.$$

If $r \equiv 1 \pmod{4}$, and $2 \in D_1$, by Lemma 3 and Lemma 4, we have

$$S(\alpha^k) \begin{cases} \neq 0, & k \in D_0 \\ \neq 0, & k \in D_1 \\ = 1, & k \in P \cup Q_2 \\ = 0, & k \in O \cup Q_3 \cup R \end{cases}.$$

Hence, $LC(s_i) = pqr - \delta$.

(ii) If $r \equiv 3 \pmod{4}$, we can prove it by using the similar method. $\square$

The result of Theorem 1 shows that the linear complexity of the binary generalized cyclotomic sequence (s_i) defined by (1) is larger than $\frac{N}{2}$, and that it can resist the Berlekamp-Massey attack.

By the definition of $\alpha, P_1, P_2, P_3, Q_2, Q_3, R$ and O , we have

$$x^p - 1 = \prod_{i \in Q_2 \cup O} (x - \alpha^i), \quad x^q - 1 = \prod_{i \in P_2 \cup O} (x - \alpha^i) \\ x^r - 1 = \prod_{i \in P_1 \cup O} (x - \alpha^i), \quad x^{pq} - 1 = \prod_{i \in R \cup P_2 \cup Q_2 \cup O} (x - \alpha^i) \\ x^{pr} - 1 = \prod_{i \in Q \cup O} (x - \alpha^i), \quad x^{qr} - 1 = \prod_{i \in P \cup O} (x - \alpha^i).$$

Suppose $d(x) = \prod_{i \in D_0 \cup D_1} (x - \alpha^i), d_j(x) = \prod_{i \in D_j} (x - \alpha^i)$ where $j = 0, 1$. Then $d(x) = d_0(x)d_1(x)$ and

$$x^N - 1 = \prod_{i=0}^{N-1} (x - \alpha^i) = \\ \frac{(x-1)(x^{pq}-1)(x^{pr}-1)(x^{qr}-1)}{(x^p-1)(x^q-1)(x^r-1)} d(x).$$

We have the following theorem.

Theorem 2 Let notations be the same as before, then the minimal polynomial $m(x)$ of the binary generalized cyclotomic sequence (s_i) satisfies

(i) If $r \equiv 1 \pmod{4}$,

$$m(x) = \begin{cases} \frac{(x^N-1)(x^p-1)^2(x^q-1)(x^r-1)}{d_0(x)(x-1)^3(x^{pr}-1)(x^{pq}-1)}, & 2 \in D_0 \\ \frac{(x^N-1)(x^p-1)^2(x^q-1)(x^r-1)}{(x-1)^3(x^{pr}-1)(x^{pq}-1)}, & 2 \in D_1 \end{cases}.$$

(ii) If $r \equiv 3 \pmod{4}$,

$$m(x) = \begin{cases} \frac{(x^N-1)(x^p-1)(x^r-1)}{d_0(x)(x-1)^2(x^{pr}-1)}, & 2 \in D_0 \\ \frac{(x^N-1)(x^p-1)(x^r-1)}{(x-1)^2(x^{pr}-1)}, & 2 \in D_1 \end{cases}.$$

Proof (i) If $r \equiv 1 \pmod{4}$, and $2 \in D_0$, we have

$$\gcd(x^N - 1, S(x)) = \frac{d_0(x)(x-1)^3(x^{pr}-1)(x^{pq}-1)}{(x^p-1)^2(x^q-1)(x^r-1)}.$$

If $r \equiv 1 \pmod{4}$, and $2 \in D_1$, we have

$$\gcd(x^N - 1, S(x)) = \frac{(x-1)^3(x^{pr}-1)(x^{pq}-1)}{(x^p-1)^2(x^q-1)(x^r-1)}.$$

(ii) If $r \equiv 3 \pmod{4}$, we can prove it by using the similar method. $\square$

4. Autocorrelation values

By the definition of x_1 and x_2 , we have $D_i \pmod{r} = D_i^{(r)}$ ($i = 0, 1$). Hence, the sequence (s_i) defined in (1) can be expressed as

$$s_i = \begin{cases} 0, & i \pmod{N} \in Q_3 \cup R \cup O \\ 1, & i \pmod{N} \in P \cup Q_2 \\ \frac{1 - \left(\frac{i}{r}\right)}{2}, & i \pmod{N} \in \mathbb{Z}_N^* \end{cases} \quad (4)$$

where $\left(\frac{\cdot}{r}\right)$ denotes the Legendre symbol. From (4) we get

$$(-1)^{s_i} = \begin{cases} 1, & i \pmod{N} \in Q_3 \cup R \cup O \\ -1, & i \pmod{N} \in P \cup Q_2 \\ \left(\frac{i}{r}\right), & i \pmod{N} \in \mathbb{Z}_N^* \end{cases}. \quad (5)$$

Lemma 5 [31] Let notations be the same as before, and $\left(\frac{\cdot}{r}\right)$ denotes the Legendre symbol, then we have

$$\sum_{x=1}^{r-1} \left(\frac{x}{r}\right) = 0$$

$$\sum_{x=0}^{r-1} \left(\frac{x}{r}\right) \left(\frac{x+a}{r}\right) = -1$$

where $\gcd(a, r) = 1$.

Theorem 3 For the generalized cyclotomic sequence (s_i) defined by (1), we have

(i) If $\tau \in P_1$,

$$N \cdot AC(s_i, \tau) = qr - pq + (p-1)(r-3) + q - 4 +$$

$$\left(\frac{\tau}{r}\right) (p-1)(q-1)(1 + (-1)^{\frac{r-1}{2}});$$

(ii) If $\tau \in P_2$,

$$N \cdot AC(s_i, \tau) = qr + (p-1)(q-4) +$$

$$(p-1)(q-2)(r-1)-4;$$

(iii) If $\tau \in P_3$,

$$\begin{aligned} N \cdot AC(s_i, \tau) &= qr - pq + 4(p-2) + q - \\ &2\left(\frac{\tau}{r}\right)(p-1)(1+(-1)^{\frac{r-1}{2}}) + \\ &\left(\frac{\tau}{r}\right)(p-1)(q-2)(1+(-1)^{\frac{r-1}{2}}); \end{aligned}$$

(iv) If $\tau \in Q_2$,

$$\begin{aligned} N \cdot AC(s_i, \tau) &= (p-2)(q-1)(r-1) + \\ &(p-4)(q+r-1); \end{aligned}$$

(v) If $\tau \in Q_3$,

$$\begin{aligned} N \cdot AC(s_i, \tau) &= (p-4)(r-4) - pq + p + \\ &\left(\frac{\tau}{r}\right)(p-2)(q-1)(1+(-1)^{\frac{r-1}{2}}); \end{aligned}$$

(vi) If $\tau \in R$,

$$\begin{aligned} N \cdot AC(s_i, \tau) &= (p-2)(q-2)r - 2(p-2) - \\ &2(q-2) - 2(r-1) + 4; \end{aligned}$$

(vii) If $\tau \in \mathbb{Z}_N^*$,

$$N \cdot AC(s_i, \tau) = 4(p-2) - 2(q-2) - 2(r-2) - pq +$$

$$\begin{aligned} &2(q-1) - 2\left(\frac{\tau}{r}\right)(p-2)(1+(-1)^{\frac{r-1}{2}}) + \\ &\left(\frac{\tau}{r}\right)(p-2)(q-2)(1+(-1)^{\frac{r-1}{2}}) + \\ &2\left(\frac{\tau}{r}\right)(1+(-1)^{\frac{r-1}{2}}). \end{aligned}$$

Proof By (5) and Lemma 5,

(i) If $\tau \in P_1$, we have

$$(-1)^{s_i+s_{i+\tau}} = \begin{cases} 1, & i \in P \setminus \{N-\tau\} \\ -1, & i \in \{0, N-\tau\} \\ -1, & i \in Q_2 \\ -1, & i \in Q_3; i+\tau \in Q_2 \\ 1, & i \in Q_3; i+\tau \in Q_3 \\ \left(\frac{\tau}{r}\right), & i \in R \\ \left(\frac{-\tau}{r}\right), & i \in \mathbb{Z}_N^*; i+\tau \in R \\ \left(\frac{i}{r}\right)\left(\frac{i+\tau}{r}\right), & i \in \mathbb{Z}_N^*; i+\tau \in \mathbb{Z}_N^* \end{cases}$$

then

$$N \cdot AC(s_i, \tau) = qr - 2 - 2 - (p-1) - (p-1) +$$

$$(p-1)(r-2) + \sum_{i \in R} \left(\frac{\tau}{r}\right) +$$

$$\sum_{\substack{i \in \mathbb{Z}_N^* \\ i+\tau \in R}} \left(\frac{-\tau}{r}\right) + \sum_{\substack{i \in \mathbb{Z}_N^* \\ i+\tau \in \mathbb{Z}_N^*}} \left(\frac{i}{r}\right) \left(\frac{i+\tau}{r}\right) =$$

$$\begin{aligned} &qr - 4 + (p-1)(r-4) + \\ &\left(\frac{\tau}{r}\right)(p-1)(q-1) + \\ &\left(\frac{-\tau}{r}\right)(p-1)(q-1) + \end{aligned}$$

$$\left(\sum_{i \in \mathbb{Z}_N} - \sum_{i \in P_1} - \sum_{i \in P_3} - \sum_{i \in Q_3} \right) \left(\frac{i}{r}\right) \left(\frac{i+\tau}{r}\right) =$$

$$\begin{aligned} &qr - 4 + (p-1)(r-4) + \\ &\left(\frac{\tau}{r}\right)(p-1)(q-1)(1+(-1)^{\frac{r-1}{2}}) - \\ &pq + 1 + (q-1) + (p-1) = \\ &qr - pq + (p-1)(r-3) + q - 4 + \\ &\left(\frac{\tau}{r}\right)(p-1)(q-1)(1+(-1)^{\frac{r-1}{2}}); \end{aligned}$$

(ii) If $\tau \in P_2$, we have

$$(-1)^{s_i+s_{i+\tau}} =$$

$$\begin{cases} 1, & i \in P \setminus \{N-\tau\} \\ -1, & i \in \{0, N-\tau\} \\ -1, & i \in Q_2 \\ \left(\frac{i}{r}\right), & i \in Q_3 \\ -1, & i \in R; i+\tau \in Q_2 \\ 1, & i \in R; i+\tau \in R \\ \left(\frac{i}{r}\right), & i \in \mathbb{Z}_N^*; i+\tau \in Q_3 \\ \left(\frac{i}{r}\right)\left(\frac{i+\tau}{r}\right) = 1, & i \in \mathbb{Z}_N^*; i+\tau \in \mathbb{Z}_N^* \end{cases}$$

then

$$N \cdot AC(s_i, \tau) = qr - 2 - 2 - (p-1) + \sum_{i \in Q_3} \left(\frac{i}{r}\right) -$$

$$\begin{aligned} &(p-1) + (p-1)(q-2) + \\ &\sum_{\substack{i \in \mathbb{Z}_N^* \\ i+\tau \in Q_3}} \left(\frac{i}{r}\right) + \sum_{\substack{i \in \mathbb{Z}_N^* \\ i+\tau \in \mathbb{Z}_N^*}} 1 = \\ &qr + (p-1)(q-4) + \\ &(p-1)(q-2)(r-1) - 4; \end{aligned}$$

(iii) If $\tau \in P_3$, we have

$$(-1)^{s_i+s_{i+\tau}} =$$

$$\begin{cases} 1, & i \in P \setminus \{N - \tau\} \\ -1, & i \in \{0, N - \tau\} \\ -\left(\frac{\tau}{r}\right), & i \in Q_2 \\ 1, & i \in Q_3; i + \tau \in R \\ \left(\frac{i + \tau}{r}\right), & i \in Q_3; i + \tau \in \mathbb{Z}_N^* \\ 1, & i \in R; i + \tau \in Q_3 \\ \left(\frac{\tau}{r}\right), & i \in R; i + \tau \in \mathbb{Z}_N^* \\ -\left(\frac{-\tau}{r}\right), & i \in \mathbb{Z}_N^*; i + \tau \in Q_2 \\ \left(\frac{i}{r}\right), & i \in \mathbb{Z}_N^*; i + \tau \in Q_3 \\ \left(\frac{-\tau}{r}\right), & i \in \mathbb{Z}_N^*; i + \tau \in R \\ \left(\frac{i}{r}\right) \left(\frac{i + \tau}{r}\right), & i \in \mathbb{Z}_N^*; i + \tau \in \mathbb{Z}_N^* \end{cases}$$

then

$$N \cdot AC(s_i, \tau) = qr - 2 - 2 - \sum_{i \in Q_2} \left(\frac{\tau}{r}\right) + (p - 1) +$$

$$\sum_{\substack{i \in Q_3 \\ i + \tau \in \mathbb{Z}_N^*}} \left(\frac{i + \tau}{r}\right) + (p - 1) + \sum_{\substack{i \in R \\ i + \tau \in \mathbb{Z}_N^*}} \left(\frac{\tau}{r}\right) -$$

$$\sum_{\substack{i \in \mathbb{Z}_N^* \\ i + \tau \in Q_2}} \left(\frac{-\tau}{r}\right) + \sum_{\substack{i \in \mathbb{Z}_N^* \\ i + \tau \in Q_3}} \left(\frac{i}{r}\right) + \sum_{\substack{i \in \mathbb{Z}_N^* \\ i + \tau \in R}} \left(\frac{-\tau}{r}\right) +$$

$$\sum_{\substack{i \in \mathbb{Z}_N^* \\ i + \tau \in \mathbb{Z}_N^*}} \left(\frac{i}{r}\right) \left(\frac{i + \tau}{r}\right) =$$

$$qr - 4 - \left(\frac{\tau}{r}\right) (p - 1) + (p - 1) +$$

$$(p - 1) \left(\sum_{i \in \mathbb{Z}_r} \left(\frac{i}{r}\right) - \left(\frac{\tau}{r}\right) \right) + (p - 1) +$$

$$\left(\frac{\tau}{r}\right) (p - 1)(q - 2) - \left(\frac{-\tau}{r}\right) (p - 1) +$$

$$(p - 1) \left(\sum_{i \in \mathbb{Z}_r} \left(\frac{i}{r}\right) - \left(\frac{-\tau}{r}\right) \right) +$$

$$\left(\frac{-\tau}{r}\right) (p - 1)(q - 2) +$$

$$\left(\sum_{i \in \mathbb{Z}_N} - \sum_{i \in P_1} - \sum_{i \in P_3} - \sum_{i \in Q_3} - \sum_{i + \tau \in Q_3} \right) \left(\frac{i}{r}\right) \left(\frac{i + \tau}{r}\right) =$$

$$qr - pq + 4(p - 2) + q -$$

$$2 \left(\frac{\tau}{r}\right) (p - 1)(1 + (-1)^{\frac{r-1}{2}}) +$$

$$\left(\frac{\tau}{r}\right) (p - 1)(q - 2)(1 + (-1)^{\frac{r-1}{2}})$$

(iv) If $\tau \in Q_2$, we have

$$(-1)^{s_i + s_{i + \tau}} =$$

$$\begin{cases} -1, & i \in P_1 \\ -1, & i \in P_2 \\ -\left(\frac{i}{r}\right), & i \in P_3 \\ 1, & i \in Q_2 \setminus \{N - \tau\} \\ -1, & i \in \{0, N - \tau\} \\ -1, & i \in Q_3; i + \tau \in P_1 \\ 1, & i \in Q_3; i + \tau \in Q_3 \\ -1, & i \in R; i + \tau \in P_2 \\ 1, & i \in R; i + \tau \in R \\ -\left(\frac{i}{r}\right), & i \in \mathbb{Z}_N^*; i + \tau \in P_3 \\ \left(\frac{i}{r}\right) \left(\frac{i + \tau}{r}\right) = 1, & i \in \mathbb{Z}_N^*; i + \tau \in \mathbb{Z}_N^* \end{cases}$$

then

$$N \cdot AC(s_i, \tau) = -(r - 1) - (q - 1) - \sum_{i \in P_3} \left(\frac{i}{r}\right) + p - 2 -$$

$$2 - (r - 1) + (p - 2)(r - 1) - (q - 1) +$$

$$(p - 2)(q - 1) - \sum_{\substack{i \in \mathbb{Z}_N^* \\ i + \tau \in P_3}} \left(\frac{i}{r}\right) + \sum_{\substack{i \in \mathbb{Z}_N^* \\ i + \tau \in \mathbb{Z}_N^*}} 1 =$$

$$(p - 2)(q - 1)(r - 1) + (p - 4)(q + r - 1);$$

(v) If $\tau \in Q_3$, we have

$$(-1)^{s_i + s_{i + \tau}} = \begin{cases} 1, & i \in P_1; i + \tau \in Q_2 \\ -1, & i \in P_1; i + \tau \in Q_3 \\ -\left(\frac{\tau}{r}\right), & i \in P_2 \\ -1, & i \in P_3; i + \tau \in R \\ -\left(\frac{i + \tau}{r}\right), & i \in P_3; i + \tau \in \mathbb{Z}_N^* \\ 1, & i \in Q_2; i + \tau \in P_1 \\ -1, & i \in Q_2; i + \tau \in Q_3 \\ -1, & i \in Q_3 \setminus \{N - \tau\}; i + \tau \in P_1 \\ -1, & i \in Q_3 \setminus \{N - \tau\}; i + \tau \in Q_2 \\ 1, & i \in Q_3 \setminus \{N - \tau\}; i + \tau \in Q_3 \\ 1, & i \in \{0, N - \tau\} \\ -1, & i \in R; i + \tau \in P_3 \end{cases}$$

$$(-1)^{s_i+s_{i+\tau}} = \begin{cases} \left(\frac{\tau}{r}\right), & i \in R; i+\tau \in \mathbb{Z}_N^* \\ -\left(\frac{-\tau}{r}\right), & i \in \mathbb{Z}_N^*; i+\tau \in P_2 \\ -\left(\frac{i}{r}\right), & i \in \mathbb{Z}_N^*; i+\tau \in P_3 \\ \left(\frac{-\tau}{r}\right), & i \in \mathbb{Z}_N^*; i+\tau \in R \\ \left(\frac{i}{r}\right)\left(\frac{i+\tau}{r}\right), & i \in \mathbb{Z}_N^*; i+\tau \in \mathbb{Z}_N^* \end{cases}$$

then

$$N \cdot AC(s_i, \tau) = 1 - (r-2) - \left(\frac{\tau}{r}\right)(q-1) -$$

$$\begin{aligned} & (q-1) - \sum_{\substack{i \in P_3 \\ i+\tau \in \mathbb{Z}_N^*}} \left(\frac{i+\tau}{r}\right) + 1 - \\ & (p-2) - (r-2) - (p-2) + \\ & (p-2)(r-2) + 2 - (q-1) + \\ & \left(\frac{\tau}{r}\right)(p-2)(q-1) - \left(\frac{-\tau}{r}\right)(q-1) - \\ & \sum_{\substack{i \in \mathbb{Z}_N^* \\ i+\tau \in P_3}} \left(\frac{i}{r}\right) + \left(\frac{-\tau}{r}\right)(p-2)(q-1) + \\ & \sum_{\substack{i \in \mathbb{Z}_N^* \\ i+\tau \in \mathbb{Z}_N^*}} \left(\frac{i}{r}\right)\left(\frac{i+\tau}{r}\right) = \end{aligned}$$

$$\begin{aligned} & (p-4)(r-4) - pq + p + \\ & \left(\frac{\tau}{r}\right)(p-2)(q-1)\left(1 + (-1)^{\frac{r-1}{2}}\right); \end{aligned}$$

(vi) If $\tau \in R$, we have

$$(-1)^{s_i+s_{i+\tau}} = \begin{cases} -\left(\frac{i}{r}\right), & i \in P_1 \\ 1, & i \in P_2; i+\tau \in Q_2 \\ -1, & i \in P_2; i+\tau \in R \\ -1, & i \in P_3; i+\tau \in Q_3 \\ -\left(\frac{i}{r}\right), & i \in P_3; i+\tau \in \mathbb{Z}_N^* \\ 1, & i \in Q_2; i+\tau \in P_2 \\ -1, & i \in Q_2; i+\tau \in R \\ -1, & i \in Q_3; i+\tau \in P_3 \\ \left(\frac{i}{r}\right), & i \in Q_3; i+\tau \in \mathbb{Z}_N^* \\ -1, & i \in R \setminus \{N-\tau\}; i+\tau \in P_2 \\ -1, & i \in R \setminus \{N-\tau\}; i+\tau \in Q_2 \\ 1, & i \in R \setminus \{N-\tau\}; i+\tau \in R \\ 1, & i \in \{0, N-\tau\} \end{cases}$$

$$(-1)^{s_i+s_{i+\tau}} = \begin{cases} -\left(\frac{i}{r}\right), & i \in \mathbb{Z}_N^*; i+\tau \in P_1 \\ -\left(\frac{i}{r}\right), & i \in \mathbb{Z}_N^*; i+\tau \in P_3 \\ \left(\frac{i}{r}\right), & i \in \mathbb{Z}_N^*; i+\tau \in Q_3 \\ 1, & i \in \mathbb{Z}_N^*; i+\tau \in \mathbb{Z}_N^* \end{cases}$$

then

$$N \cdot AC(s_i, \tau) = - \sum_{i \in P_1} \left(\frac{i}{r}\right) + 1 - (q-2) - (r-1) -$$

$$\sum_{\substack{i \in P_3 \\ i+\tau \in \mathbb{Z}_N^*}} \left(\frac{i}{r}\right) + 1 - (p-2) - (r-1) +$$

$$\sum_{\substack{i \in Q_3 \\ i+\tau \in \mathbb{Z}_N^*}} \left(\frac{i}{r}\right) - (q-2) - (p-2) +$$

$$(p-2)(q-2) + 2 - \sum_{\substack{i \in \mathbb{Z}_N^* \\ i+\tau \in P_1}} \left(\frac{i}{r}\right) -$$

$$\sum_{\substack{i \in \mathbb{Z}_N^* \\ i+\tau \in P_3}} \left(\frac{i}{r}\right) + \sum_{\substack{i \in \mathbb{Z}_N^* \\ i+\tau \in Q_3}} \left(\frac{i}{r}\right) + \sum_{\substack{i \in \mathbb{Z}_N^* \\ i+\tau \in \mathbb{Z}_N^*}} 1 =$$

$$\begin{aligned} & (p-2)(q-2)r - 2(p-2) - \\ & 2(q-2) - 2(r-1) + 4; \end{aligned}$$

(vii) If $\tau \in \mathbb{Z}_N^*$, we have

$$(-1)^{s_i+s_{i+\tau}} =$$

$$\begin{cases} -1, & i \in P_1; i+\tau \in R \\ -\left(\frac{i+\tau}{r}\right), & i \in P_1; i+\tau \in \mathbb{Z}_N^* \\ -1, & i \in P_2; i+\tau \in Q_3 \\ -\left(\frac{\tau}{r}\right), & i \in P_2; i+\tau \in \mathbb{Z}_N^* \\ 1, & i \in P_3; i+\tau \in Q_2 \\ -1, & i \in P_3; i+\tau \in Q_3 \\ -1, & i \in P_3; i+\tau \in R \\ -\left(\frac{i+\tau}{r}\right), & i \in P_3; i+\tau \in \mathbb{Z}_N^* \\ 1, & i \in Q_2; i+\tau \in P_3 \\ -\left(\frac{\tau}{r}\right), & i \in Q_2; i+\tau \in \mathbb{Z}_N^* \\ -1, & i \in Q_3; i+\tau \in P_2 \\ -1, & i \in Q_3; i+\tau \in P_3 \\ 1, & i \in Q_3; i+\tau \in R \\ \left(\frac{i+\tau}{r}\right), & i \in Q_3; i+\tau \in \mathbb{Z}_N^* \\ -1, & i \in R; i+\tau \in P_1 \\ -1, & i \in R; i+\tau \in P_3 \\ 1, & i \in R; i+\tau \in Q_3 \end{cases}$$

$$(-1)^{s_i+s_{i+\tau}} = \begin{cases} \left(\frac{\tau}{r}\right), & i \in R; i+\tau \in \mathbb{Z}_N^* \\ -\left(\frac{i}{r}\right), & i \in \mathbb{Z}_N^* \setminus \{N-\tau\}; i+\tau \in P_1 \\ -\left(\frac{-\tau}{r}\right), & i \in \mathbb{Z}_N^* \setminus \{N-\tau\}; i+\tau \in P_2 \\ -\left(\frac{i}{r}\right), & i \in \mathbb{Z}_N^* \setminus \{N-\tau\}; i+\tau \in P_3 \\ -\left(\frac{-\tau}{r}\right), & i \in \mathbb{Z}_N^* \setminus \{N-\tau\}; i+\tau \in Q_2 \\ \left(\frac{i}{r}\right), & i \in \mathbb{Z}_N^* \setminus \{N-\tau\}; i+\tau \in Q_3 \\ \left(\frac{-\tau}{r}\right), & i \in \mathbb{Z}_N^* \setminus \{N-\tau\}; i+\tau \in R \\ \left(\frac{i}{r}\right)\left(\frac{i+\tau}{r}\right), & i \in \mathbb{Z}_N^* \setminus \{N-\tau\}; i+\tau \in \mathbb{Z}_N^* \\ \left(\frac{\tau}{r}\right), & i=0 \\ \left(\frac{-\tau}{r}\right), & i=N-\tau \end{cases}$$

then

$$N \cdot AC(s_i, \tau) = -1 - \sum_{\substack{i \in P_1 \\ i+\tau \in \mathbb{Z}_N^*}} \left(\frac{i+\tau}{r}\right) - 1 - \left(\frac{\tau}{r}\right)(q-2) +$$

$$1 - (r-2) - (q-2) - \sum_{\substack{i \in P_3 \\ i+\tau \in \mathbb{Z}_N^*}} \left(\frac{i+\tau}{r}\right) + 1 -$$

$$\left(\frac{\tau}{r}\right)(p-2) - 1 - (r-2) + (p-2) +$$

$$\sum_{\substack{i \in Q_3 \\ i+\tau \in \mathbb{Z}_N^*}} \left(\frac{i+\tau}{r}\right) - 1 - (q-2) + (p-2) +$$

$$\left(\frac{\tau}{r}\right)(p-2)(q-2) - \sum_{\substack{i \in \mathbb{Z}_N^* \setminus \{N-\tau\} \\ i+\tau \in P_1}} \left(\frac{i}{r}\right) -$$

$$\left(\frac{-\tau}{r}\right)(q-2) - \sum_{\substack{i \in \mathbb{Z}_N^* \setminus \{N-\tau\} \\ i+\tau \in P_3}} \left(\frac{i}{r}\right) -$$

$$\left(\frac{-\tau}{r}\right)(p-2) + \sum_{\substack{i \in \mathbb{Z}_N^* \setminus \{N-\tau\} \\ i+\tau \in Q_3}} \left(\frac{i}{r}\right) +$$

$$\left(\frac{-\tau}{r}\right)(p-2)(q-2) +$$

$$\sum_{\substack{i \in \mathbb{Z}_N^* \setminus \{N-\tau\} \\ i+\tau \in \mathbb{Z}_N^*}} \left(\frac{i}{r}\right) \left(\frac{i+\tau}{r}\right) + \left(\frac{\tau}{r}\right) + \left(\frac{-\tau}{r}\right) =$$

$$4(p-2) - 2(q-2) - 2(r-2) - pq +$$

$$2(q-1) - 2\left(\frac{\tau}{r}\right)(p-2)(1+(-1)^{\frac{r-1}{2}}) +$$

$$\left(\frac{\tau}{r}\right)(p-2)(q-2)(1+(-1)^{\frac{r-1}{2}}) +$$

$$2\left(\frac{\tau}{r}\right)(1+(-1)^{\frac{r-1}{2}})$$

and the result follows. $\square$

Theorem 4 Let notations be the same as before.

(i) If $p \bmod r \in D_0^{(r)}, \tau \in P_{30}$, or $p \bmod r \in D_1^{(r)}, \tau \in P_{31}$, we have $\left(\frac{\tau}{r}\right) = 1$.

If $p \bmod r \in D_0^{(r)}, \tau \in P_{31}$, or $p \bmod r \in D_1^{(r)}, \tau \in P_{30}$, we have $\left(\frac{\tau}{r}\right) = -1$.

(ii) If $q \bmod r \in D_0^{(r)}, \tau \in Q_{30}$, or $q \bmod r \in D_1^{(r)}, \tau \in Q_{31}$, we have $\left(\frac{\tau}{r}\right) = 1$.

If $q \bmod r \in D_0^{(r)}, \tau \in Q_{31}$, or $q \bmod r \in D_1^{(r)}, \tau \in Q_{30}$, we have $\left(\frac{\tau}{r}\right) = -1$.

(iii) If $pq \bmod r \in D_0^{(r)}, \tau \in P_{10}$, or $pq \bmod r \in D_1^{(r)}, \tau \in P_{11}$, we have $\left(\frac{\tau}{r}\right) = 1$.

If $pq \bmod r \in D_0^{(r)}, \tau \in P_{11}$, or $pq \bmod r \in D_1^{(r)}, \tau \in P_{10}$, we have $\left(\frac{\tau}{r}\right) = -1$.

(iv) If $\tau \in D_0$, we have $\left(\frac{\tau}{r}\right) = 1$.

If $\tau \in D_1$, we have $\left(\frac{\tau}{r}\right) = -1$.

From Theorem 4, the exact value of the autocorrelation function of the new binary generalized cyclotomic sequences (s_i) can be determined. By Theorem 3 and Theorem 4, if $\tau \in P_1$, the autocorrelation values of (s_i) satisfy $\frac{pr+qr-3pq}{pqr} \leq AC(s_i, \tau) \leq \frac{pr+qr+pq}{pqr}$, which is close to 0 if r is a big prime. If $\tau \in P_2$, the autocorrelation values of (s_i) satisfy $AC(s_i, \tau) \leq 1$. In other cases, we can use the same method as above to determine the autocorrelation values which is close to 0 or 1.

Table 1 shows the difference between the proposed sequences and the sequences in [27]–[29]. $LC_{\max}$ is the maximum value that the linear complexity of the sequence can achieve, $\times$ indicates the autocorrelation of the sequences were not provided in the corresponding studies.

Table 1 Difference between our sequences and the sequences in [27]–[29]

Sequence	Balance	$LC_{\max}$	AC
[27]	$ (p-1)(q+r-1)-qr+2 $	$pqr-(p-1)r-1$	$\times$
[28]	$ (p-1)(q+r-3)-(q-1)(r-1)-1 $	$pqr-(p-1)(q-1)-1$	$\times$
[29]	$ (p_1-1)(p_2+p_3-1)-p_2p_3+2 $	$p_1p_2p_3-p_1$	$\times$
Ours	$ (p-1)(q+r-3)-qr+2 $	$pqr-(p-1)(r-1)-1$	$\checkmark$

5. Simulations

Based on the tools C++ and Matlab, Table 2 lists the exact linear complexities of (s_i) by the Berlekamp-Massey algorithm and linear complexities of (s_i) given by Theorem 1. The conclusion about the linear complexity of (s_i) in Theorem 1 is correct. Table 3 and Table 4 give the exact autocorrelation values of (s_i) by the definition of the autocorrelation function $AC(s_i, \tau)$ and autocorrelation values of (s_i) given by Theorem 3 when $r \equiv 3 \pmod 4$ and $r \equiv 1 \pmod 4$, respectively. The autocorrelation values of (s_i) in Theorem 3 are the same as the tested values. Fig. 1 and Fig. 2 show that the autocorrelation values of (s_i) when $p = 19$, are $q = 11, r = 3$ and $p = 19, q = 11, r = 5$, respectively.

Table 2 Linear complexity of the new sequences (s_i)

(p, q, r)	g	$2 \in D_1$	t	$LC(s_i)$	Our results
(19,11,3)	2	Yes	627	590	590
(19,11,5)	2	Yes	1 045	792	792
(47,31,23)	11	No	33 511	17 318	17 318
(43,31,17)	3	No	22 661	10 648	10 648

Table 3 Autocorrelation values of (s_i)

$(479,383,227)$	$AC(s_i, \tau)$	Our results
$\tau \in P_1$	0.000 263	0.000 263
$\tau \in P_2$	0.994 766	0.994 766
$\tau \in P_3$	-0.002 263	-0.002 263
$\tau \in Q_2$	0.995 795	0.995 795
$\tau \in Q_3$	-0.001 850	-0.001 850
$\tau \in R$	0.990 573	0.990 573
$\tau \in \mathbb{Z}_N^*$	-0.004 370	-0.004 370

Fig. 1 Autocorrelation values of $p = 479, q = 383, r = 227$

Table 2 shows that the linear complexity of (s_i) is high when $2 \in D_1$. By Table 3 and Table 4, the autocorrelation value of the new sequences is close to 1 when

$\tau \in P_2 \cup Q_2 \cup R$, and close to 0 in other cases. The cardinality of $P_2 \cup Q_2 \cup R$ approximates pq ; therefore, the autocorrelation values are close to 0 with a probability of approximately $\left(1 - \frac{1}{r}\right)$. Thus, when r is a big prime, the new sequence has a good autocorrelation.

Table 4 Autocorrelation values of (s_i)

$(167,83,41)$	$AC(s_i, \tau)$	Our results
$\tau \in P_1, \left(\frac{\tau}{r}\right) = 1$	0.040 741	0.040 741
$\tau \in P_1, \left(\frac{\tau}{r}\right) = -1$	-0.055 068	-0.055 068
$\tau \in P_2$	0.975 457	0.975 457
$\tau \in P_3, \left(\frac{\tau}{r}\right) = 1$	0.029 057	0.029 057
$\tau \in P_3, \left(\frac{\tau}{r}\right) = -1$	-0.063 246	-0.063 246
$\tau \in Q_2$	0.987 591	0.987 591
$\tau \in Q_3, \left(\frac{\tau}{r}\right) = 1$	0.034 132	0.034 132
$\tau \in Q_3, \left(\frac{\tau}{r}\right) = -1$	-0.061 100	-0.061 100
$\tau \in R$	0.963 217	0.963 217
$\tau \in D_0$	0.022 518	0.022 518
$\tau \in D_1$	-0.069 243	-0.069 243

Fig. 2 Autocorrelation values of $p = 167, q = 83, r = 41$

6. Conclusions

In this paper, we determine the linear complexity and the autocorrelation values of the new binary generalized cyclotomic sequences (s_i) . The results show that the linear complexity of the sequence is high when $2 \in D_1$. Furthermore, the autocorrelation of the sequence is close to 1 when $\tau \in P_2 \cup Q_2 \cup R$, and the autocorrelation is close to 0 at other points. Therefore, the new sequence has a good autocorrelation when r is a big prime.

References

- [1] TURYN R J. The linear generation of the legendre sequence. *Journal of the Society for Industrial and Applied Mathematics*, 1964, 12(1): 115–116.
- [2] DING C, HESSESETH T, SHAN W. On the linear complexity of Legendre sequences. *IEEE Trans. on Information Theory*, 1998, 44(3): 1276–1278.
- [3] STANTON R G, SPROTT D A. A family of difference sets. *Canadian Journal of Mathematics*, 1958, 6(1): 73–77.
- [4] DING C. Binary cyclotomic generators. *Lecture Notes in Computer Science*, 1994, 1008: 29–60.
- [5] DING C. Linear complexity of generalized cyclotomic binary sequences of order 2. *Finite Fields and Their Applications*, 1997, 3(2): 159–174.
- [6] DING C. Autocorrelation values of generalized cyclotomic sequences of order two. *IEEE Trans. on Information Theory*, 1998, 44(4): 1699–1702.
- [7] BAI E, FU X, XIAO G. On the linear complexity of generalized cyclotomic sequences of order four over $\mathbb{Z}_{pq}$. *IEICE Trans. on Fundamentals*, 2005, 88(1): 392–395.
- [8] BRANDSTÄTTER N, WINTERHOF A. Some notes of the two-prime generator of order 2. *IEEE Trans. on Information Theory*, 2005, 51(10): 3654–3657.
- [9] DING C, HELLESETH T. New generalized cyclotomy and its application. *Finite Fields and Their Applications*, 1998, 4(2): 140–166.
- [10] BAI E, LIU X. Generalized cyclotomic sequences of order four over $\mathbb{Z}_{pq}$ and their autocorrelation values. *Chinese Journal of Engineering Mathematics*, 2008, 25(5): 894–900. (In Chinese)
- [11] BAI E, LIU X, XIAO G. Linear complexity of new generalized cyclotomic sequences of order two of length pq . *IEEE Trans. on Information Theory*, 2005, 51(5): 1849–1853.
- [12] YAN T, HONG L, XIAO G. The linear complexity of new generalized cyclotomic binary sequences of order four. *Information Sciences*, 2008, 178(3): 807–815.
- [13] JIN S, KIM Y, SONG H. Autocorrelation of new generalized cyclotomic sequences of period p^n . *IEICE Trans. on Fundamentals*, 2010, 93(11): 2345–2348.
- [14] KE P, ZHANG J, ZHANG S. On the linear complexity and the autocorrelation of generalized cyclotomic binary sequences of length $2p^m$. *Designs, Codes and Cryptography*, 2013, 67(3): 325–339.
- [15] LI S, CHEN Z, SUN R, et al. On the randomness of generalized cyclotomic sequences of order two and length pq . *IEICE Trans. on Fundamentals*, 2007, 90(9): 2037–2041.
- [16] LI S, CHEN Z, FU X, et al. Autocorrelation values of new generalized cyclotomic sequences of order two and length pq . *Journal of Computer Science and Technology*, 2007, 22(6): 830–834.
- [17] Meidl W. Remarks on a cyclotomic sequence. *Designs, Codes and Cryptography*, 2009, 51(1): 33–43.
- [18] YAN T, SUN R, XIAO G. Autocorrelation and linear complexity of the new generalized cyclotomic sequences. *IEICE Trans. on Fundamentals*, 2008, 90(4): 857–864.
- [19] YAN T, DU X, XIAO G, et al. Linear complexity of binary Whiteman generalized cyclotomic sequences of order $2k$. *Information Sciences*, 2009, 179(7): 1019–1023.
- [20] YAN T, HUANG B, XIAO G. Cryptographic properties of some binary generalized cyclotomic sequences with the length p^2 . *Information Sciences*, 2008, 178(4): 1078–1086.
- [21] YAN T, LI S, XIAO G. On the linear complexity of generalized cyclotomic sequences with the period p^m . *Applied Mathematics Letters*, 2008, 21(2): 187–193.
- [22] YAN T, CHEN Z, XIAO G. Linear complexity of Ding generalized cyclotomic sequences. *Journal of Shanghai University*, 2007, 11(1): 22–26.
- [23] ZHANG J, ZHAO C, MA X. Linear complexity of generalized cyclotomic binary sequences of length $2p^m$. *Applicable Algebra in Engineering, Communication and Computing*, 2010, 21(2): 93–108.
- [24] CHEN Z, DU X, WU C. Pseudo-randomness of certain sequences of k symbols with length pq . *Journal of Computer Science and Technology*, 2011, 26(2): 276–282.
- [25] CHEN Z, DU X. Linear complexity and auto-correlation values of a polyphase generalized cyclotomic sequence of length pq . *Frontiers of Computer Science*, 2010, 4(4): 529–535.
- [26] DING C, HELLESETH T. On cyclotomic generator of order r . *Information Processing Letters*, 1998, 66(1): 21–25.
- [27] CHANG Z, ZHOU Y, KE P. Linear complexity of new generalized cyclotomic sequences of order two and length pqr . *Acta Electronica Science*, 2015, 43(1): 166–170. (In Chinese)
- [28] LIU L, YANG X, DU X, et al. On the linear complexity of new generalized cyclotomic binary sequences of order two and period pqr . *Tsinghua Science and Technology*, 2016, 21(3): 295–301.
- [29] LYU C, XIAO G Z, YAN T J. On linear complexity of a new generalized cyclotomic sequence of order two over $\mathbb{Z}_{p_1 p_2 p_3}$. *Journal of Computational Information Systems*, 2015, 11(17): 6355–6362.
- [30] CUSICK T W, DING C, RENVALL A. *Stream cipher and number theory*. Amsterdam: North-Holland, 1988.
- [31] LIDL R, NIEDERREITER H. *Finite fields*. 2nd ed. Cambridge: Cambridge University Press, 1997.

Biographies

JIA Wenjuan was born in 1994. She received her B.S. degree from Northwest Normal University. She is a Ph.D candidate in cryptography from Xidian University. Her research interests include Lattice-based cryptography, stream cipher and pseudorandom sequence.
E-mail: 18368910175@163.com

GAO Juntao was born in 1979. He received his D.E. degree in cryptography from Xidian University. He is now an associate professor. His research interests include block chain, stream cipher and pseudorandom sequence.
E-mail: jtgao@mail.xidian.edu.cn

ZHANG Peng was born in 1992. He received his B.S. degree from Henan University and his M.S. degree in cryptography from Xidian University. His research interests include compressive sensing and cryptography.
E-mail: 297966738@qq.com